Investigation 1.1

Problem Solving in Geometry

Problems to Get You Started

These are not five-minute problems. Each investigation will take time.

The following problems were selected to give you a glimpse of some of the ideas, techniques, styles, and languages that you will work with throughout the modules. They are from different areas of geometry, and the connections between them may not be apparent right away. As your studies progress, try to notice where these problems fit in. Some problems may use unfamiliar vocabulary. Keep track of new words and ideas in a journal, and find out what they mean and how they fit together.

Geometry Around You

1. **Write and Reflect** What shape are manhole covers? Think of a good reason for choosing *that* shape instead of all others. Explain your reason in writing.

2. **Write and Reflect** What shape are the nuts on fire hydrants? Why do you think *that* shape was selected? Write as complete an explanation as you can.

On your own, list at least five things that are cylindrical. When you are done, go on to the discussion with your group.

For Discussion

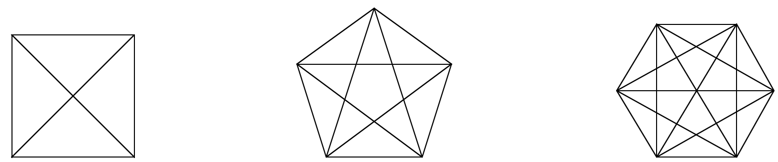
Compare all the lists of cylindrical objects in your group. You need to agree on a list of ten objects that are cylinders. The goal is to have as many objects as possible on your list that are not on any other group's list.

3. **The Handshake Problem** If everyone in your class shook hands just once with everyone else, how many handshakes would that be?

If *penta-*, *hexa-*, *hepta-*, and *octa-* mean 5, 6, 7, and 8, and *poly-* means "many," what does *-gon* mean?

4. **Diagonals in Regular Polygons** How many diagonals are there in a square? In a pentagon? In a hexagon? In a heptagon? In an octagon? Write a rule that would enable you to determine the number of diagonals in *any* regular polygon, given the number of sides.

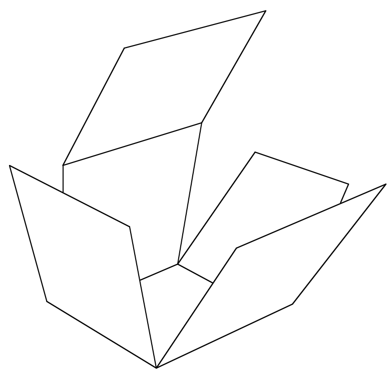
Why is a square not a "tetragon" and a triangle not a "trigon"? By the way, *dia-gonal* has *-gon* in it, too! The *dia-* part means "across" or "spanning."



NETS

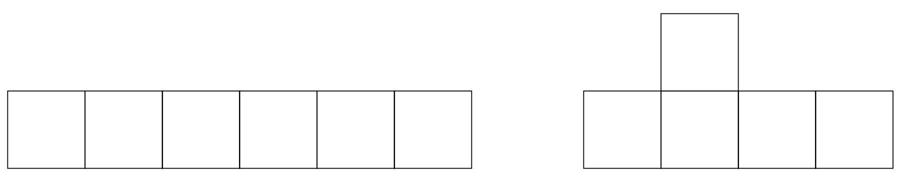
What is a *face* of a cube?

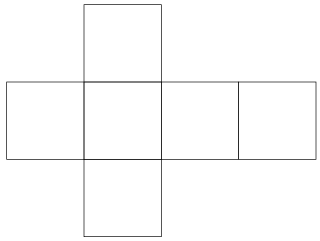
Imagine unfolding a cube, so that all of its faces are laid out as a set of squares attached at their edges. The resulting diagram is called a *net* for a cube. There are many nets for a cube.

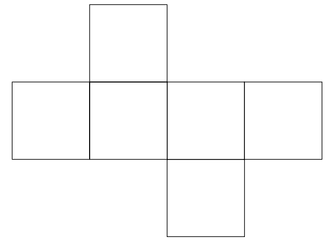


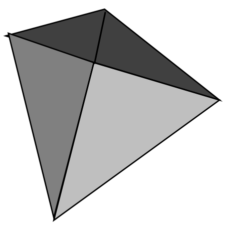
Continue to unfold this in your mind.

5. Which of the following are nets for cubes? Describe your strategy for deciding.









A *tetrahedron* is a three-dimensional shape made from four triangular faces.

6. How many *different* nets for a cube can you make? Describe how you thought about the problem, what method you used to generate different nets, and how you checked whether a new one really was different.

7. Find all the possible nets for a regular tetrahedron.

THE "TRIANGLE INEQUALITY"

For this experiment, you need three dice and eighteen rods—three each of lengths 1 unit through 6 units. Roll the dice and pick three corresponding rods. For example, if you roll 5, 3, 5, pick two rods of size 5 and one of size 3. Try to make a triangle using the three rods as sides of the triangle. Some sets of three rods will work, and others won't.

8. Repeat the experiment several times—rolling the dice, picking the rods, and trying to make a triangle—and keep a table of your results. For the combinations that don't work, write an explanation of what went wrong when you tried to make a triangle.

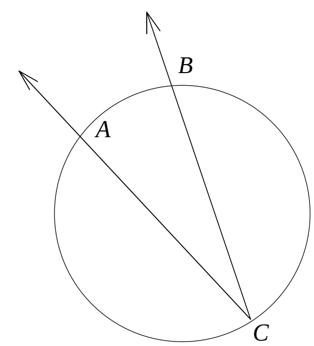
9. Write a rule that will tell you whether three given rods will make a triangle.

10. **Write and Reflect** Your experiments dealt only with sidelengths from 1 to 6, and not with noninteger lengths, like $4\frac{1}{2}$ or 3.14159. Write a rule that explains how you can tell if *any* three segments will actually fit together to make a triangle. Some sets of three lengths just don't work. Explain why, and how to predict this from the lengths involved.

ANGLES INSCRIBED IN SEMICIRCLES

Many important results in geometry came about because someone noticed an *invariant*: something about a situation that stays the same while other parts of the situation vary. In this problem, you will look for an invariant related to inscribed angles.

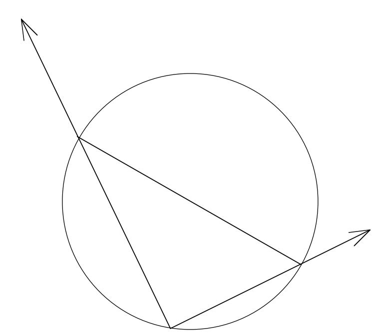
An angle *inscribed in a circle* has a vertex on the circle, and sides that go through two other points on the circle.



Angle ACB is an inscribed angle.

If you have a circle, you can make a semicircle by drawing a diameter across the circle. How do you make sure the line segment going across is a diameter?

To make an angle *inscribed in a semicircle*, the sides need to go through the endpoints of a diameter.



An angle inscribed in a semicircle

The more accurately you make your drawings, the more confidence you can place in your experimental results.

11. Draw a semicircle. Then inscribe an angle in that semicircle. What is the measure of that angle? Inscribe another angle in your semicircle. Measure it. What varies? What remains the same?

12. Draw two different diameters in one circle. Connect the endpoints of the diameters to make a four-sided figure. What kind of figure is it? Repeat the experiment with a new pair of diameters. What is invariant here? Explain.

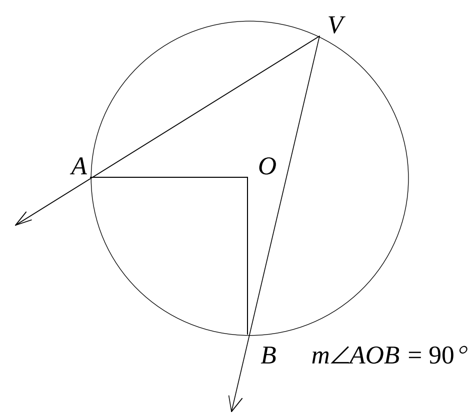
13. A *very* important habit of mind is to look for connections. How are Problems 11 and 12 related?

14. In Problem 11, you inscribed an angle in a semicircle. What can you say about an angle inscribed in a quarter circle? In a three-quarter circle? Can you make a general rule?

Going the long way on the circle from point *A* to point *B* is $\frac{3}{4}$ of the way around the circle. $\angle AVB$ is *inscribed* in an arc that is $\frac{3}{4}$ of the circle and "contains" an arc that is $\frac{1}{4}$ of the circle.

$\angle AOB$ is also called the *central angle* of the arc from *A* to *B*.

"$m\angle AOB$" is read "the measure of angle *AOB*."

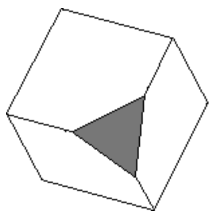


An angle inscribed in a three-quarter circle

15. **Write and Reflect** If you inscribed a hundred angles in a semicircle and they all had the same measure, would that guarantee that *every* angle inscribed in a semicircle would be that size? Why?

CROSS SECTIONS

This slice through a cube shows one possible cross section of a cube.



A cross section is the face you get when you make one slice through an object. The cross section shown is a triangular cross section, but for simplicity we call it a triangle.

These problems ask you to visualize the slices through solid objects.

16. What cross sections can you make by slicing a cube? Record which of the shapes below you are able to create, and describe how you did it.

- **a.** A square
- **b.** An equilateral triangle
- **c.** A rectangle that is not a square
- **d.** A triangle that is not equilateral
- **e.** A pentagon
- **f.** A hexagon
- **g.** An octagon
- **h.** A parallelogram that is not a rectangle
- **i.** A trapezoid

17. Can you create any shapes that are not listed above? Draw and name any other cross sections you can make.

18. If you think that any of the shapes on the list in Problem 16 are *impossible* to make by slicing a cube, explain what makes them impossible.

19. What cross sections can you get from a sphere?

20. What cross sections can you get from a cylinder?

CHECKPOINT

21. Illustrate and briefly explain each of these terms:

cross section

diagonal

diameter

net

semicircle

regular polygon

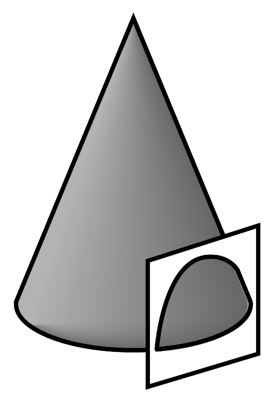
inscribed

TAKE IT FURTHER

22. Draw a circle and then draw a square that is inscribed in the circle. Explain how you know that it is a square.

A cone has a circular base and a point at the vertex. Think of an ice cream cone.

23. Here is a drawing to help you picture a cone. What cross sections can you make by slicing a cone? Make some sample cones and cut through them at different angles.



Cross sections of a cone are called *conic sections*. They are very important in mathematics and physics. You will undoubtedly see them again.

A pyramid has a polygonal base and triangular sides. A square pyramid has a square for its base.

24. What cross sections can you make by slicing a square pyramid? Make some sample square pyramids and cut through them at different angles.

25. Try constructing quadrilaterals from rods of different lengths. What rules can you find about the lengths that can and can't be used in constructing four-sided figures?

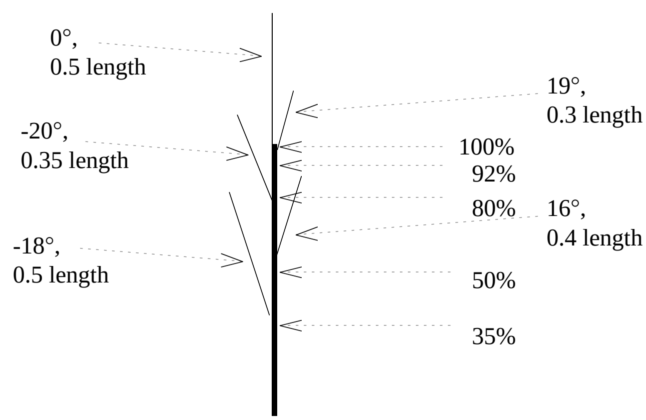
AND WHAT IS GEOMETRY?

Simply put, geometry is the attempt to understand space, shape, and dimension. Parts of "geo-metry"—earth-measure—grew out of the age-old interests of explorers to map where they had been, and of landowners to determine the boundaries of their holdings. Other parts were invented by artists, who wished to portray convincingly what they saw with their eyes or in their minds, and by inventors and engineers who wished to make devices that would fit together and work. Geometric ideas have also come from the needs of architects and builders, whose work must be both strong and beautiful, and from surveyors, planners, and builders, who must guarantee that tunnels or railroad tracks built from two directions will actually meet precisely as planned.

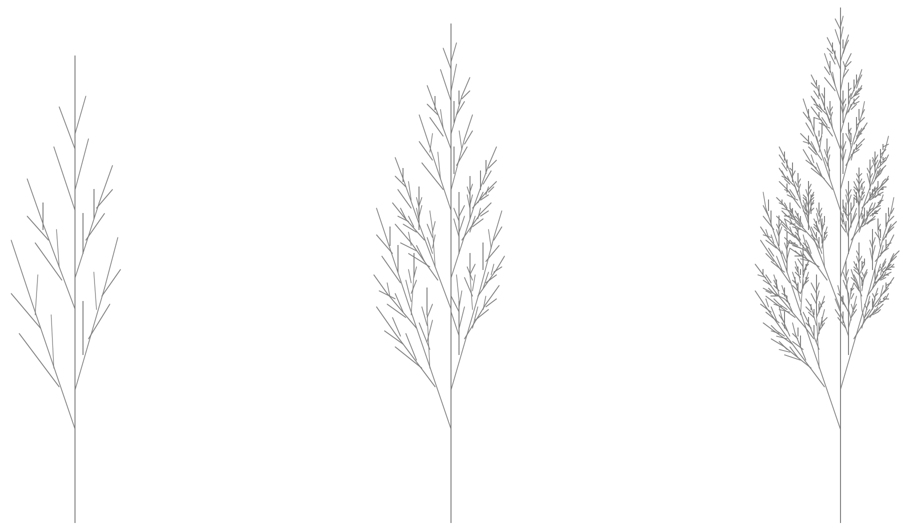
Fractal geometry is a relatively-new branch of geometry named just 30 years ago and studied in depth only since that time.

Geometric "shapes" are not only the spartan shapes with names like *square* or *pyramid*. Fractal geometry can describe many intricate shapes like tree-shapes or cloud-shapes with surprisingly simple methods.

The following picture describes the rule for building the trees shown after Problem 26.



26. Decide how to interpret this pictorial "rule." How many limbs does it show growing from the trunk? How far up the trunk is the lowest limb? How long is that limb, compared with the trunk? What angle does that limb make with the trunk? How far up the trunk is the highest limb? How do its length and angle compare with those of the trunk?



The picture on the left shows what happens when each limb sprouts five branches according to the same rules—the same relative distance along the branch, size relationships, and angles of growth. Apply the same rules again to grow five twigs on each branch; the result looks like the middle picture. The last picture, with five leaves per twig, is very tree-like.

Mathematics is sometimes pictured as an isolated field, unrelated to other subjects, fields, or hobbies. And geometry is often presented as unrelated even to other kinds of mathematics. Historically, none of this was the case. Within mathematics, geometry has been a source of great insight into other mathematical domains, like algebra, and vice versa. Some of the most important mathematicians were also artists, scientists, inventors, clerics, or combined these careers. Art, mathematics, science, and social sciences are still closely related. Cognitive science, one of the fastest growing scientific fields, combines the study of mathematics, psychology, computer science, and biology. Artist M.C. Escher (1898–1972) was fascinated with mathematical ideas and incorporated many into his drawings. Dancer and choreographer Michael Moschen uses ideas from mathematics and physics to inspire his remarkable art.

Examples abound of mathematics influencing other fields and vice versa, but these often are neglected when subjects are taught separately. The *Connected Geometry* modules are designed to help you find these connections or invent them for yourself, building links with your other interests and studies, and within mathematics itself.

PERSPECTIVE ON HOW GEOMETRY HAS CHANGED IN THE LAST 3000 YEARS

The study of geometry has traditionally been based on Euclid's work. This essay describes how geometry has evolved since Euclid's time and how it continues to develop and change today.

The most famous geometry books of all time, *The Elements*, were written in Egypt about 2300 years ago. Their author, Euclid, compiled and systematized in them his own ideas and all the geometric ideas that he had known, many of which had been developed by the mathematicians who lived before him.

But not all mathematics, including geometry, is thousands of years old. It is not all known, or all finished with nothing new to discover. It should be no surprise that a lot of new geometry has been invented since Euclid. After all, 2300 years is a long time. Art, architecture, mechanical design, clothing design, navigation, communication, and other fields have all changed and have raised new geometric questions.

In fact, geometry has changed enormously in the last *twenty* years, and it continues to change rapidly today. New applications, especially involving computers, have expanded the scope of what is possible to explore in geometry.

Computer graphics and animation have created new jobs and demanded new research. New geometric techniques have been developed to solve problems of optimizing paths—for example, finding the most efficient routes for snowplows, the best routes for airlines, the least expensive network of telephone wires, or the smallest microchip. The mathematical study of knots, which started in support of a defunct theory held by chemists a century ago, has recently become a powerful tool in modern physics and biochemistry. Astrophysicists have new notions about the shape of space and have invented new mathematical tools like "optical geometry." The advent of the computer age has made possible mathematical modeling of complex natural phenomena like weather, and has given rise to a new branch of mathematics called dynamical systems. And there is combinatorial geometry, algebraic geometry, differential geometry, ...

Some of the geometry you will learn predates Euclid. But you will also learn about geometry that is being researched and discovered today. What ties Euclid to modern geometers, and high school students to professional mathematicians, is the use of the same powerful mathematical habits of mind. These habits will become a central focus of your work as you learn to visualize new situations, to experiment with fresh ideas, and to prove your own conjectures.

Investigation 1.2

PICTURING AND DRAWING

MAKING MENTAL PICTURES

Many problems are solved (or made easier) by making pictures. The pictures can be on paper, on a computer, or in your head.

Even when your goal is to make a picture on paper, mental pictures are important: practice in making clear and detailed pictures in your head can help you to draw better.

1. Sometimes, you make mental pictures because the things that you need to see are not where you can see them. Answer these questions without moving from where you are.

a. How many windows are in your home?

b. Is the bathroom doorknob in your home on the right or left as you enter?

c. Where did you find your shoes this morning?

d. Think of someone you saw this morning (but can't see now). What color clothing was he or she wearing?

e. How many outside doors are in your school?

When you need to see things with your mind, it often helps to close your eyes so that you aren't distracted by what you see with your eyes.

2. Sometimes, you make mental pictures because the things that you need to see are abstract *ideas*. Picture a point hanging in midair somewhere in the room. Picture a second point somewhere else.

a. How many different *straight lines* pass through those two points?

b. How many different *circles* pass through those two points?

c. Are there squares that can be built using those two points as corners? How many can you think of?

d. How many cubes of different sizes are there that have the two points as corners?

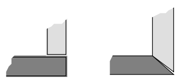
Mental objects are not always abstract things like points and lines. Sometimes they are ideas for very concrete things. If you are planning to make something—whether it's a shirt, a shelf, or a house—you must "see" it *very* clearly before it exists. Otherwise, the parts are not likely to fit together.

How will the sides fit together? You can pick a "simple" or a "fancy" way,

3. Design a simple box—a bottom and four sides—to be made of wood. Decide on dimensions for the outside (or inside) of the box, and then figure out the exact

size of each of the five pieces of wood. Remember that wood has thickness ($\frac{3}{4}''$ is common, but you can choose what you want) and that the sides have to be attached to each other and to the bottom, presumably with nails or screws. (*Build* the box if you have the materials.)

but be sure that your dimensions allow your choice to work.



Designing a sleeve for a shirt is much more difficult than designing a box. For one thing, a shoulder is a less "regular" shape than a box. Also, to cut the right shape, one needs to know how the "tube" looks while it is still a flat piece of cloth.

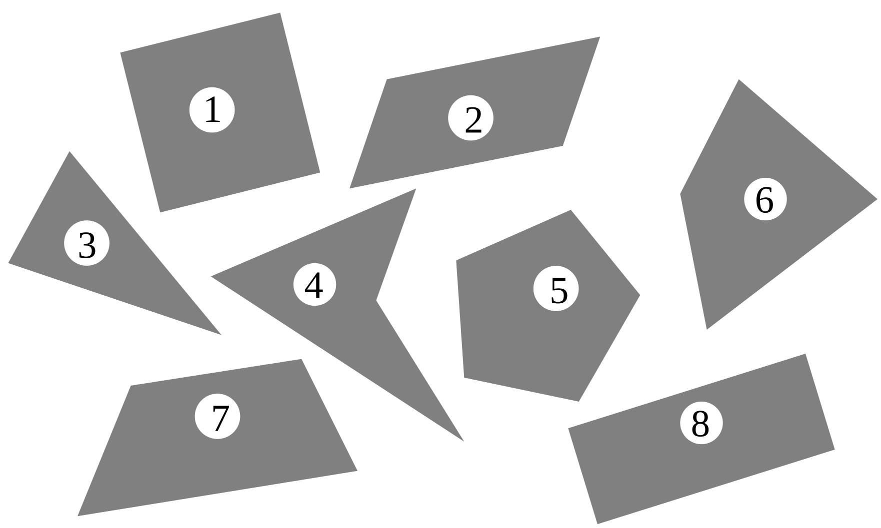
4. Figure out and draw the shape to cut from flat cloth in order to make a shirtsleeve for yourself. (The shape is almost certainly *not* a rectangle.) If you can get cloth to experiment with, test your pattern.

One mathematician wrote: "Sleeves were the bane of my mercifully-brief sewing career." Almost anyone not skilled in clothing design finds this very frustrating. Just *trying* to do it gives one a sense of the remarkable understanding of shape and space that a clothing designer must develop! Artists, carpenters, and mechanics must develop similar skills.

Designing things that fit together at angles often requires that we be able to picture cross sections of solids. And, for artists to make convincing drawings of scenes that they have pictured in their imagination, they must be able to draw the correct shape of a shadow.

5. a. Imagine a square casting a shadow on a flat floor or wall. Can the shadow be nonsquare? Can the shadow be nonrectangular? (That is, can the angles in the shadow ever vary from 90°?)

b. Which of the following shapes could cast a square shadow and which could not? Explain your answers.



6. What shadow-shapes can an equilateral triangle cast? What shadow-shapes can a circle cast?

Pictures drawn on paper or by computer can help when the images with which you are experimenting become too complicated to keep in your head. For the next problem, try to notice when you prefer mental pictures and when you prefer paper ones.

Of course, there are many ways to write a letter. T, for example, can look like *T* or T or 𝔗 or *t* or . . . . For this problem, *you* can decide how *your* letters will look, but keep them simple.

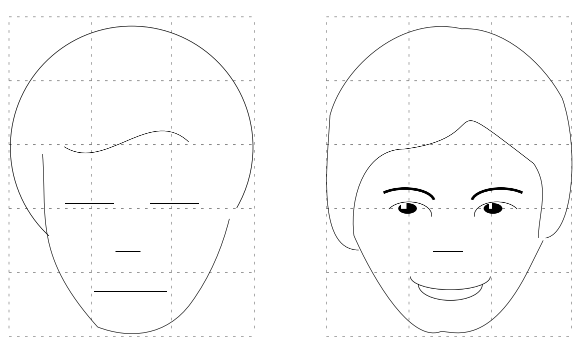
7. Some letters of the alphabet can be cut into two parts that are pretty much the same. Some cannot.

a. The letter **T** can be cut into two matching parts in two different ways. A slice can separate the top from the stem, making two straight "sticks" of roughly the same size. Also, a vertical slice through the stem can separate the T into two upside-down L-shaped pieces. Draw pictures to illustrate these two different methods.

b. Which other letters can be sliced into two matching parts by one straight cut?

ANALYZING SHAPES GEOMETRICALLY

Analyzing complex scenes geometrically can often help you see (and sketch them) more clearly. For example, here are two sets of "guide lines" for a drawing of a face. The first one shows few details: the "construction lines" simply show the rough sizes and placement of the features. The second set adds some detail.



Plans for a drawing of a face

8. Describe the size and placement relationships that these plans show for this face. For example, the mouth is about one third the width of the head.

If you practice seeing things in this very geometric way, it can improve your drawings greatly!

9. Draw rough guide lines (as on the previous page, on the left) for a drawing of a friend's face. Notice where the corners of the mouth are located with respect to the eyes; get the mouth and nose guide lines roughly the right fraction of the way from chin to eyes. Attend closely to proportions, but don't worry about fractions of an inch. (For example, are your friend's eyes roughly halfway from top to bottom or more like two thirds of the way?)

The Catalan artist Salvador Dali created a wonderful and easily recognizable portrait of Abraham Lincoln using only the geometry of Lincoln's face without any of the actual details. You may be able to find a copy of this painting in an art book in your library.

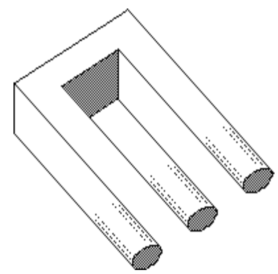
10. Repeat Problem 9 for a famous face like Eleanor Roosevelt's or Albert Einstein's or Martin Luther King's or the Mona Lisa's. Sketch rough guide lines only—no shading or fancy details. Do your guide lines make the face recognizable?

ANALYZING THREE-DIMENSIONAL SCENES GEOMETRICALLY

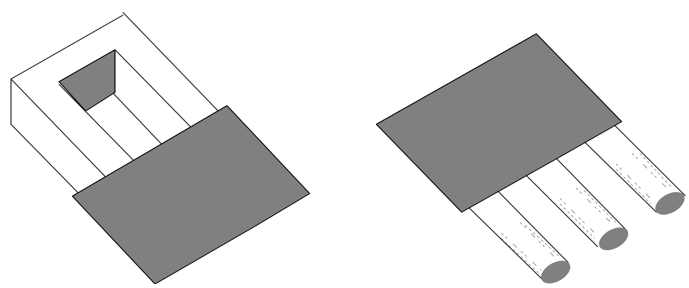
It is especially important to pay attention to the geometry when you are trying to draw three-dimensional figures. Some of the figures you will draw in this activity are "impossible" figures, pictures that confuse the eye. Why impossible figures? Why not just ordinary things? Partly for fun, but mostly because these strange pictures *do* confuse the eye. In order to draw them, you must take them apart in your mind to see how they are made up. And when you do, you will understand what it is about their geometry that makes them "impossible"; you will see them differently, and you will be able to draw them.

11. **An Impossible Picture** Look at this drawing. It's a strange thing. Or is it a thing at all?

a. What do you see? Explain why the drawing looks "off."



b. When the drawing is partially covered (see below), what's left doesn't look nearly so peculiar. Why does covering part of the picture help?



You may find making a sketch very hard at first, because it's so difficult to picture this thing in your head. It may help to cover all but the part you are working on.

c. Sketch your own copy of the original drawing.

d. Now, close the book and don't look at your copy of the picture. Try to draw this thing from memory. Explain how you drew the picture.

This picture is hard to draw not *just* because this is a trick picture, but also because it suggests a three-dimensional (3D) object.

DRAWING 3D

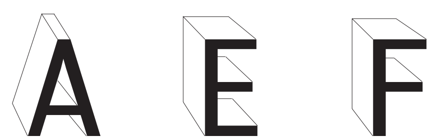
When a "flat" shape is a polygon, then the solid created by translating that base shape into the third dimension is called a "prism." Some of the base shapes you might use (like the letter C) are not polygons, and so the solids based on them are not typically called prisms. The authors of this book call them "quasiprisms." You and your classmates might want to invent a name of your own.

At times, you've probably tried to draw your name (or someone else's) in letters that have a 3D look.

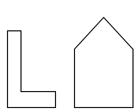
Below is a recipe for turning "flat" letters like these:

A E F

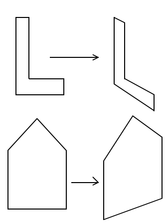
into "solid" letters like these:



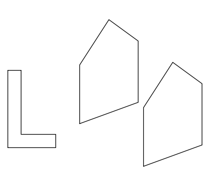
Follow the steps as they are written. This recipe uses special terms—like prism, parallel, and line segment—so that it will apply to *any* drawing you make.



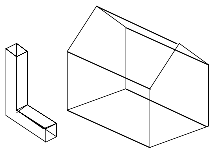
Step 1: Choose a letter or other shape and draw it in your notebook. The shape you have drawn is the "base" of the prism that you will draw. The two examples shown here begin with an L-shaped hexagon and a house-shaped pentagon.



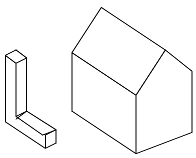
Step 2 (Optional): If you like, you can rotate the flat shape into the third dimension. Vertical lines stay vertical. Parallel lines stay parallel. Other changes you "make by eye." You'll get good with practice.



Step 3: A prism needs two parallel bases, so draw a second base shape near the first. Take care to make the line segments of the second base parallel to (and the same size as) the corresponding line segments in the first base.



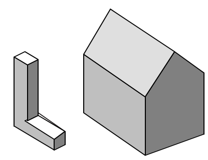
Step 4: Now connect the corresponding corners of the two bases. This is called a "wire frame drawing."



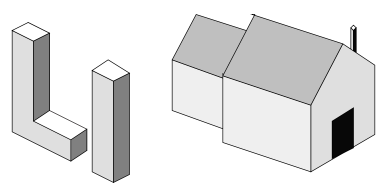
Step 5: Wire frame pictures can be visually confusing. Hiding (erasing) the back lines helps the eye make sense out of the pictures. (It is usually easier to start with the wire frame and then erase than it is to draw the correct view "from scratch.") Hide the appropriate lines in your picture.

Now that the pentagonal prism looks like a house, the word "base" is especially confusing. The *house*'s "base" is a rectangle sitting on the ground. The *prism* is "based on" a pentagon: Its bases are the front and back of the "house."

Step 6: Shading also helps the eye make sense of a picture. Faces that are parallel to each other should, in general, be shaded the same way.



Let your imagination go. Add some details, but keep true to the rules so that the picture continues to look right.



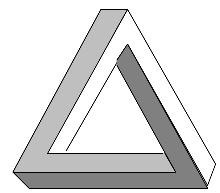
In *true perspective* drawings, the connecting lines are *not* parallel, and the rules are more complicated.

12. **Write and Reflect** The lines that connect the two bases in this kind of wire frame drawing will be parallel to each other. First convince yourself that this is true, and then find a convincing way to explain *why*.

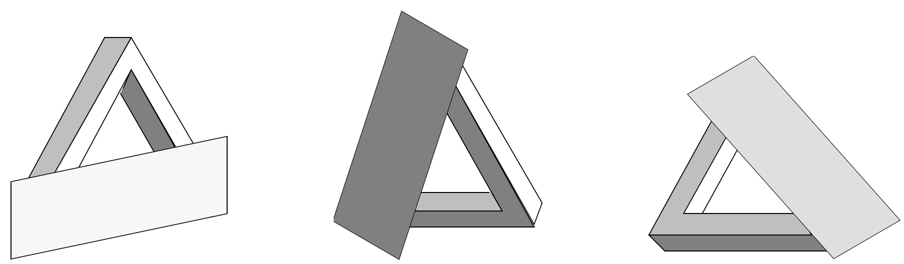
13. **a.** In the letters or shapes you drew, which faces are "parallel faces"?

b. Why do you think parallel faces *would* normally be shaded the same way? (Under what conditions would they be shaded differently?)

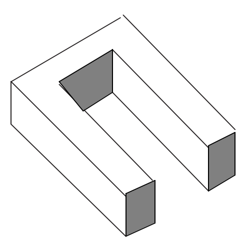
14. **Another Impossible Picture** Here is another "impossible 3D figure" to draw. Study it. Cover parts if this would help. Then *try to draw it without any parts covered*. Include the shading. (What information does the shading give?)

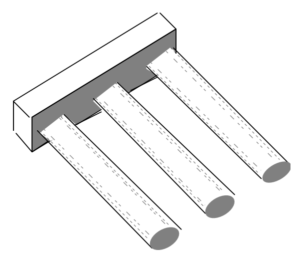


15. Nothing appears "wrong" about the three figures below, and yet each is just the figure above with a single bar covered. On the basis of these figures (or using some other insight of your own), explain what is "wrong" with the original figure shown in Problem 14.

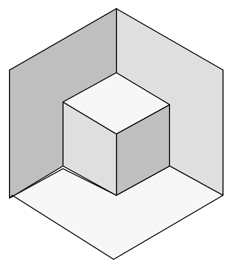


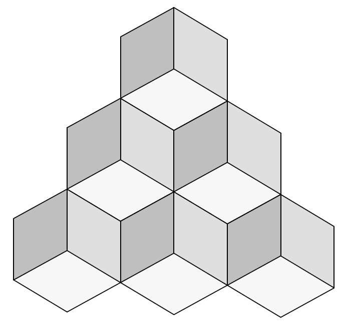
16. You've seen the next figures before—almost. Draw them, first from the sketches, and then from memory. The task is somewhat different from the task of drawing their impossible cousin. Be sure to include the shading.





17. Figures that can be seen in more than one way play tricks on the eye, similar to the tricks the impossible figures play. Try drawing these. What confuses the eye this time?

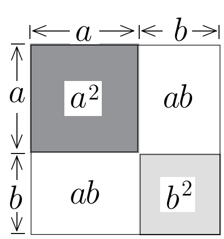




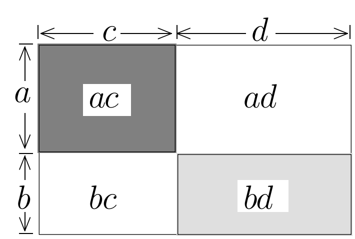
USING PICTURES TO EXPLAIN IDEAS

Although pictures *are* shapes in space, they don't have to be *about* shape or space. Sometimes pictures help us "visualize" quantities, or relationships among quantities—things that are not really visual at all.

18. Figure out, and then explain in words, what the pictures below can tell you about the multiplication of binomials.

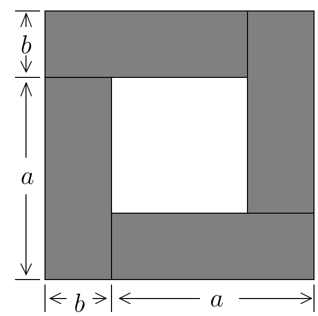


$(a+b)^2 = a^2 + 2ab + b^2$



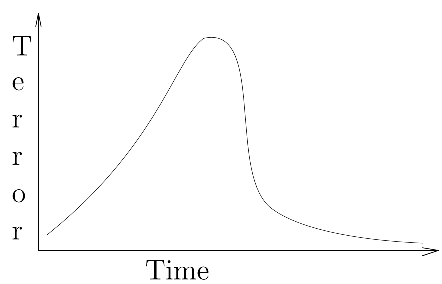
$(a+b)(c+d) = ac + ad + bc + bd$

19. Explain how this picture says the same thing that the algebraic equation does.



$$(a+b)^2 - 4ab = (a-b)^2$$

20. In describing the feelings she had about tests and examinations when she was a student, a mathematics teacher once sketched this picture. Explain in words what her picture is saying.



21. Sketch a graph of your own, showing your energy over the course of a day, or perhaps how hungry you feel as the hours pass, or . . . .

CHECKPOINT

22. In a short amount of time, you've run across many special terms. Look back through your work and list the terms you've used in your writing.

23. Look back through the reading and the problems and list geometric and mathematical terms that are new to you. Find a definition for any about which you are not certain.

24. Find a drawing of a prism in a book, magazine, or newspaper. Does it seem to be drawn using the method shown in "Drawing 3D"? Describe the similarities and differences between the printed prism and the one drawn by the method given here.

25. Can you make a triangle with the following sidelengths? (Try to answer without building the triangle.)

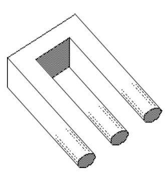
a. $2''$, $7''$, and $5''$

b. $3''$, $7''$, and $5''$

c. $2''$, $7''$, and $3''$

d. $4''$, $4''$, and $4''$

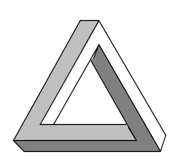
26. Make a sketch that illustrates the equation $d(c + f) = dc + df$.



27. On your drawing of the impossible "pronged" figure, locate and label an example of as many of the following as you can find.

plane	face
line segment	edge
line	vertex
cylinder	midpoint
circle	right angle
perpendicular planes	perpendicular segments
intersection of planes	intersection of segments

28. How many different planes are pictured in this sketch? (How many planes are hidden?)

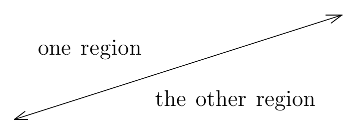


29. Choose a letter and draw a $2''$-tall 3D version of it using the method shown in "Drawing 3D." Shade in a base of the prism (or quasiprism). Explain why that shape is a "base."

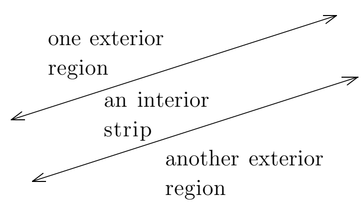
TAKE IT FURTHER

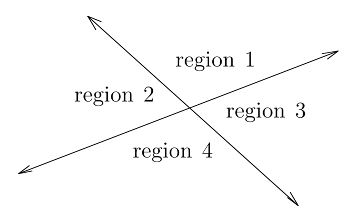
What does “infinite” mean?

30. A plane is infinite. One (infinite) line on it will divide the plane into two regions.



Two lines may cut the plane into three or four regions, depending how they are placed.





What is the maximum number of plane regions that you can produce with five lines?

This is the 3D version of Problem 30, but it is hard to picture. We’ve known people to puzzle over this for weeks.

31. Space is infinite. One (infinite) plane through it will divide it into two regions. Two planes may cut space into three or four regions, depending how they are placed. With three planes, it is possible to cut space into as few as four and as many as eight regions. What is the maximum number of regions that you can produce with four planes? With five planes?

32. There is also a 1D version of Problem 30. A line is infinite. One point on it will divide it into two regions. Into how many regions will two points divide it? Is there more than one possible answer? What is the maximum number of regions that you can produce with five points?

WAYS TO THINK ABOUT IT

Compare dividing a line (Problem 32) to dividing a plane (Problem 30). What was the same? What changed? How can you use this reasoning to help you extend the idea to dividing space (Problem 31)?

See Problem 9 in this investigation to remind yourself about guide lines.

33. How could you use guide lines to make a good enlargement or reduction of a picture of a face? Take one of your pictures from Problem 9 or 10 of this section and make a face that is twice as large. Write a paragraph to explain how you did it.

34. Take a guide line plan of a face and sketch it onto a grid that has been stretched in one direction. Compare the stretched version with the original.

Escher often talked of his naiveté about mathematics. However, he corresponded with mathematicians about his ideas, read mathematical papers on tiling to get more ideas, and even contributed ideas to mathematics.

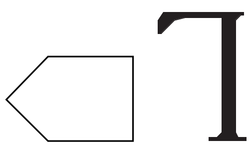
35. **Project** Find out about the life and work of various artists who have used and contributed to geometric ideas. For example, learn about M.C. Escher's "impossible" pictures, and why and how he drew them. Attempts to explore "impossible geometry" have not been limited to drawing and painting: photographers have also contributed their ideas and skills by creating "impossible photographs." Many artists, like Vasarely and Mondrian, have used geometric ideas as the basis for much of their art. Dali often played with the perspective or other aspects of the geometry or topology of his subjects. And the less well-known artist Tchelitchew added more and more geometric abstraction to his drawings and paintings over the course of his career.

Investigation 1.3

DRAWING AND DESCRIBING SHAPES

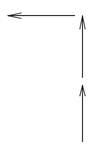
It is often quite important to be able to describe shape and other spatial information accurately with words.

There are many ways that shapes can be described.



Names Some shapes have special names, like *circle* or *square*. A name, then, may be enough to describe your shape. When shapes are *like* other things that you can name, you may need some extra words: “like an upside-down L,” “like a house lying on its side,” “saddle-shaped”. . . .

Features Often the shape is not directly known, and it must be discovered from some set of features. Scientists face this situation when they try to deduce the shape of a molecule from what they know about the atoms that make it up, or from the way it scatters light or X-rays.



Recipes Sometimes, saying what a picture looks like is not as helpful as saying how to make it. For example, when giving traveling directions, you’d be much more likely to say “walk two blocks, turn left, and walk another block” than to describe the path as “an upside-down L.” The recipes we use in mathematics are often called *algorithms* or *constructions*.

“One picture is worth more than a thousand words.” –Chinese Proverb

You will practice *all* these ways so that you will be prepared to combine them to best suit your purposes.

As you describe pictures (or draw pictures from descriptions) in this investigation, try to notice when you’re thinking about *names*, *features*, and *recipes*.

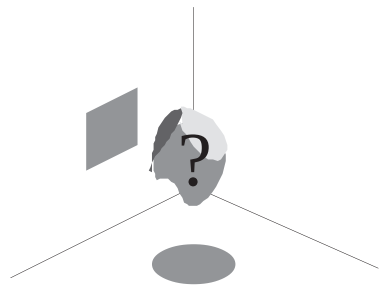
For most objects, the shape of a shadow depends on how the object is lit. The next few problems ask you to think about shapes and the shadows they cast, and to *deduce* properties of a shape based on its shadows.

Direct sunlight on a clear day generally creates the most-distinct and least-distorted shadows.

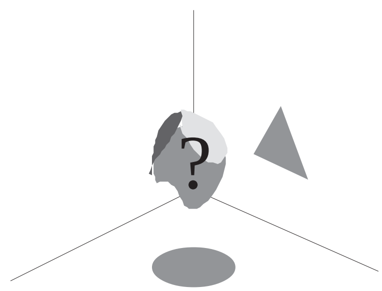
1. What shapes are possible for the shadow of a soup can? Of a cube?

2. A solid object casts a circular shadow on the floor. When it’s lit from the front, it casts a square shadow on the back wall. What shape might it be? Try to make

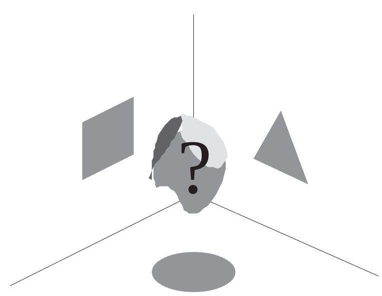
such a shape out of clay, sponge, dough, or other material. Describe it in words as well as you can; try to draw a picture of it.



3. A solid object casts a circular shadow on the floor. When it's lit from the left, it casts a triangular shadow on the right wall. What shape might it be? Try to make such a shape out of clay, sponge, dough, or other material. Describe it in words as well as you can; try to draw a picture of it.



4. Suppose an object casts a circular shadow on the floor, a triangular shadow when lit from the left, *and* a square shadow when lit from the front. What shape might the object be? Try to make one out of clay, sponge, dough, or other material. Describe it in words as well as you can; try to draw a picture of it.



5. Here are some walking directions. If you followed these directions, what shape would your path be? What direction would you be facing at the end?

> Face north. Walk four feet. Turn right. Walk six feet. Turn right again. Walk four feet. Turn right again, walk six feet. Turn right again.

"Turn right" doesn't say how far to turn, and yet you probably have made an assumption about it. What was your assumption? What makes it seem reasonable? *Must* it be correct?

6. In the last two problems, when were you thinking *Names*, when were you thinking *Features*, and when were you thinking *Recipes*?

Investigation 1.4

Drawing from a Recipe

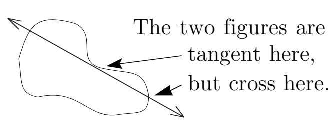
In this investigation, you will translate words into drawings, and drawings into words. As you work, you will encounter more ideas and terminology needed for talking about geometry. The directions are intended to be clear and unambiguous, but you may need to figure out some things that are unfamiliar to you.

Words and Pictures

A picture is worth 202 words. Here's proof! If you follow the instructions in Problem 1 carefully, you'll get a picture.

A geometric line is infinite. We can't draw it, so we draw a stand-in for what we imagine.

Tangent means *just touching*.



1. **a.** Draw a horizontal line (well, line segment).

b. Draw two same-size circles above that line and tangent to it. Leave some space between the two circles—roughly as much space as the diameter of the circles.

c. Draw a line segment above the two circles and tangent to them. It should be just long enough to extend slightly beyond the two circles. Label this segment's left endpoint L and its right endpoint R.

d. At L, draw a segment about half the length of $\overline{LR}$, extending *up* from L and perpendicular to $\overline{LR}$. Label its top endpoint B. Draw an identical vertical segment up from R, and label its top endpoint F.

Two points define a segment, ray, or line. The symbol over the two letters tells whether a segment $\overline{AB}$, a ray $\overrightarrow{AB}$, or a line $\overleftrightarrow{AB}$ is being named.

e. Draw segment $\overline{BF}$.

f. With a very light pencil line (which you will later erase), extend segment $\overline{BF}$ about two thirds of its length to the right. Label the new endpoint X.

g. Lightly sketch a perpendicular down from X, making it roughly the same length as $\overline{FR}$. Find the *midpoint* of this new segment and label it M.

h. Draw $\overline{MR}$. Now you can erase the construction lines that you drew in steps 1f and 1g. They are no longer needed. What is your picture?

How did the authors know the lines would meet at the same point in *your* drawing? It's just how triangles behave. Later you'll find your own explanation for *why* it works.

2. Carefully and precisely draw a large triangle. Find the midpoint of each side. Connect each midpoint to the opposite vertex. Label the point where all three connecting lines cross.

The next three problems give geometric directions for drawing letters of the alphabet. The directions are "fairly good," but you will sometimes have to make guesses about what is meant, and check if the results make sense.

***Equi-*, 'equal.' *Lateral*, 'side.' *Equilateral*, 'Equal sides.'**

3. Draw a roughly equilateral triangle, two inches on a side, with a horizontal base. Find and connect the midpoints of the two nonhorizontal sides. Erase the base of the triangle. What letter did you draw?

4. Draw a circle with a half-inch radius. Draw a slightly larger circle directly below the first and tangent to it (just touching it). Lightly sketch a vertical construction line through the centers of both circles. Lightly sketch two horizontal construction lines, one through the center of each circle. In the top circle, erase the 90° arc in the bottom right quadrant. In the bottom circle, erase the 90° arc in the top left quadrant. Then erase the construction lines. What letter did you draw?

What meaning do the words *quadrant*, *quadrilateral*, *quarter*, and the Spanish word *cuatro* have in common?

The whole circle is 360° of arc, so a 90° arc is a quarter of the circle.

5. Make a circle and divide it into 90°-arcs with two diameters that are about 45° from vertical and perpendicular to each other. Erase the arc on the right side of the circle, and then erase the diameters. What letter did you draw?

6. Write directions for drawing each initial of your name. Use precise language. Some letters are complicated, so take advantage of any terms from geometry that will help make your directions clear.

TAKE IT FURTHER

7. Your teacher will pass out pictures.

   a. Figure out how to draw each picture you are given. Then write directions so someone else could draw it without seeing the original picture.

   b. Exchange your directions with a partner to test them out. Draw nothing more and nothing less than what is written in the directions. Do you get the same pictures that your partner started with? Did your partner get the same picture that you started with?

Investigation 1.5

Constructing from Features: Problem Solving

Some drawings can *not* be made accurately with the tools available.

In the previous investigation, you were given recipes for a drawing but weren't told what the drawing was. Life more often presents problems the other way around: you know what's needed, but not how to make it. Such problems have always inspired creative, inventive thinking. The solution—the how-to-make-it part—always depends on what tools are available.

For example, suppose you are allowed one ruler, one piece of paper, and one pencil, and your task is to draw a triangle whose sides are 3″, 4″, and 6″.

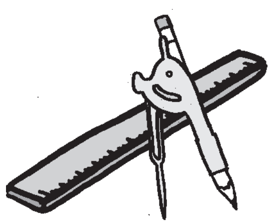
"It is said that geometry is the art of applying good reasoning to bad drawings." Henri Poincaré

1. Try it. Using only one ruler, one piece of paper, one pencil, and no other tools or aids, find a method for making a picture of a triangle whose sides are 3″, 4″, and 6″. Describe your construction method.

Drawings are *aids to problem-solving*; constructions are *solutions to problems*.

Geometers distinguish between a *drawing* and a *construction*. Drawings are intended to aid memory, thinking, or communication, and they don't have to be much more than rough sketches to serve this purpose quite well. The essential element of a construction is that it is a kind of guaranteed recipe—it shows how, in principle, a figure can be accurately drawn with a specified set of tools. Your *method* for Problem 1 is a construction. The picture that you make just illustrates your construction.

Hand Construction Tools



A ruler is a straightedge with special marks on it.

In your study of geometry, you will probably use both hand construction tools and computer tools. The computer tools will be introduced later.

Compass A compass is any device—even a suitably rigged piece of string—that allows you to move a pencil around any point you pick, and keeps the pencil at a fixed distance from that point. This property not only allows you to construct "circles of any size . . . placed anywhere," but also to copy distances.

Straightedge Any straight edge—even the edge of a piece of paper—can help you draw a straight line segment. When something is called a straightedge, it generally means it is unmarked, and cannot be used to measure distances. Straightedges are just to allow one to draw "a straight line (segment) . . . as long as one needs."

Measuring Devices A ruler can measure the length of a segment or the distance between two points. A protractor or goniometer measures angles.

The *Connected Geometry* module *The Cutting Edge* uses dissection to develop and prove many theorems and area formulas.

Paper Paper is not just something on which to write and draw. The symmetries created by folding it can be used in very creative ways to construct geometric figures. And dissection—cutting paper figures and rearranging the parts—can be a powerful aid to reasoning.

String With string and tacks, you can easily build devices to construct circles, ellipses, spirals, and other curves.

CONSTRUCTING TRIANGLES

For these problems, use whatever hand construction tools seem best. Keep track not only of your answer, but *how* you solved the problem—what tools you used, and what you did with them.

Watch out, though! The fact that you've read a *description* of a shape doesn't mean that the shape *exists*. If it doesn't exist, there's no way to draw it.

2. If possible, construct a triangle with sides of the given lengths.

- **a.** 3″, 5″, and 7″
- **b.** 3″, 5″, and 4″
- **c.** 3″, 8″, and 4″
- **d.** 2″, 3″, and 3″

3. For each triangle you constructed in Problem 2:

- **a.** Measure the angles of your triangle.
- **b.** Compare your results with someone else's results for the same triangle. Are the two triangles identical? Do the angles of the two triangles match exactly?
- **c.** Summarize and explain what you observed.

4. Sum the three angles in each triangle you drew for Problem 2. Is the result invariant?

FOR DISCUSSION

You may have found an invariant in Problem 4, but you only tested a few triangles. Do you believe that your invariant will hold for all triangles? For only some triangles? Which ones? What would it take to convince you that no matter what kind of triangle you have, the sum of the angles will be the same?

5. If possible, construct a triangle with the given angles.

a. 40°, 60°, and 80°

b. 60°, 70°, and 80°

c. 30°, 60°, and 90°

d. 90°, 90°, and 90°

e. 120°, 30°, and 30°

6. For each triangle you constructed in Problem 5:

a. Measure the sides (in inches or centimeters, whichever seems more convenient).

b. Find the ratio of the longest side to the shortest side (divide longest by shortest).

In one of the triangles you made, two sides were the same length. What do you think *caused* that?

c. Compare the two results with someone else's results for the same triangle. Are the two triangles (more or less) identical? Are the ratios (more or less) identical?

d. Summarize and explain what you observed.

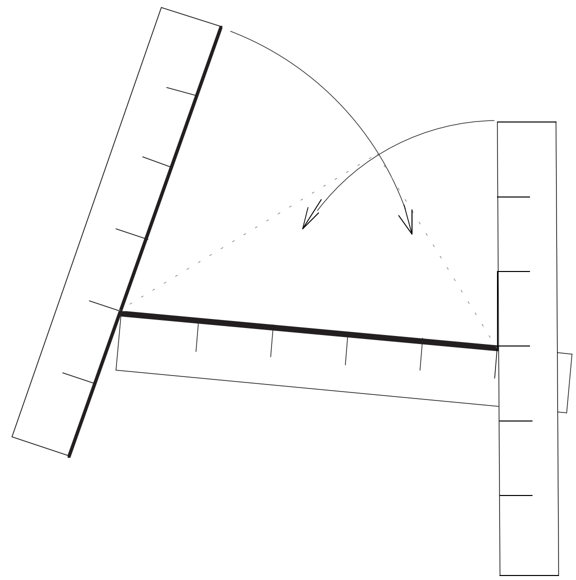
7. In Problems 6c and 6d, you compared triangles that different people constructed with three given angles. In Problems 3b and 3c, you did the same sort of thing for triangles constructed from three given sides. Now compare the two experiments. In what ways are the results different?

8. Take one of your triangles from Problem 2. Without measuring anything, make a new triangle with sides half as long as the sides of your original triangle.

COMPASSES

Here is a method that students have used to draw a triangle when all three sidelengths are known. It will be illustrated using the sidelengths 3″, 4″, and 5″.

Draw a segment to have one of the given lengths. (In the picture below, the drawn segment is the 5″ segment, and the ruler that measured it is still showing.) Then, using two rulers, . . .



9. Look carefully at the picture and figure out what the students did to construct the 3″, 4″, 5″ triangle. Then, complete the description of their method, stating carefully and precisely how each ruler was used.

10. Describe how to adapt this method to work with only *one* ruler.

FOR DISCUSSION

This section is called "Compasses," but they haven't been mentioned yet. What does this ruler trick have to do with compasses? How can compasses help you in geometric constructions?

ANGLES AND CIRCLES

11. Construct two circles, each of which passes through the other's center.

12. **a.** Construct an *obtuse triangle*—a triangle that has one angle larger than 90°. Label the vertex at the obtuse angle A and label the other two vertices B and C. Construct a circle whose diameter is the segment $\overline{BC}$. The circle will pass through vertices C and B. Is vertex A inside, outside, or on the circle?

b. Construct an *acute triangle*—a triangle in which all angles are smaller than 90°. Label its vertices D, E and F. Construct a circle whose diameter is the segment $\overline{DE}$. Is vertex F inside, outside, or on the circle?

c. Construct a *right triangle*—a triangle that has one angle exactly 90°. Label the vertex at the right angle G and label the other two vertices H and I. Construct a circle with $\overline{HI}$ as its diameter. Is vertex G inside, outside, or on the circle?

FOR DISCUSSION

A tentative conclusion is called a "conjecture."

Compare your results for Problem 12 with those of your classmates. Summarize your observations. What tentative conclusions can you make? What other problems or facts that you have encountered in this module relate to this one?

13. Which of these four-sided figures are possible?

a. A rectangle in which a diagonal is twice the length of one of the sides

b. A rectangle in which a diagonal is the same length as one of the sides

c. A quadrilateral whose diagonals are perpendicular to each other

d. A rectangle whose diagonals are perpendicular to each other

14. **Challenge** Construct a quadrilateral with at least one 60° angle and sides that are all the same length.

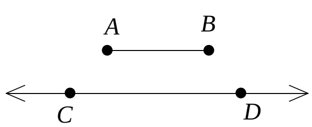
OTHER CONSTRUCTIONS

Use any hand construction tools you want, *except rulers*. No segment measurement is allowed. Don't forget paper folding.

"Answers" to the following construction problems are descriptions of how you made the constructions, along with the pictures you made. Some are harder than others; if you have difficulties with one, try another. Later, when you return to the hard ones, you'll have more experience and knowledge.

15. Draw a line segment on your paper. Without measuring anything, construct its midpoint.

A line goes on forever, so you can just draw a piece (segment) of one. One convention for distinguishing a line from a segment in drawings is to extend the line past the points that name it, and to attach arrowheads.



$\overleftrightarrow{CD}$ is a line and $\overline{AB}$ is a segment.

16. Draw a line on your paper. Then

a. construct a line that is perpendicular to it;

b. construct a line that is parallel to it.

17. Start with an ordinary sheet of $8\frac{1}{2}'' \times 11''$ paper. Using just folding and scissors, create the largest square you can which, itself, contains no cuts.

18. Start with the square you constructed in Problem 17. (You may want to make a few of them.) Then

a. construct a square with exactly one fourth the area of your original square;

b. construct a square with exactly one half the area of your original square.

19. Without measuring, construct two rectangles that you are sure have the same area but do not have the same sidelengths.

20. Draw an angle on your paper. Construct its bisector. (An angle bisector is a ray that cuts the angle exactly in half, making two equal-size angles).

21. For each construction, start with a (freshly-drawn) segment. Then construct each of the following shapes.

a. An isosceles triangle with your segment as one of the two equal-length sides

b. An isosceles triangle whose base is your segment

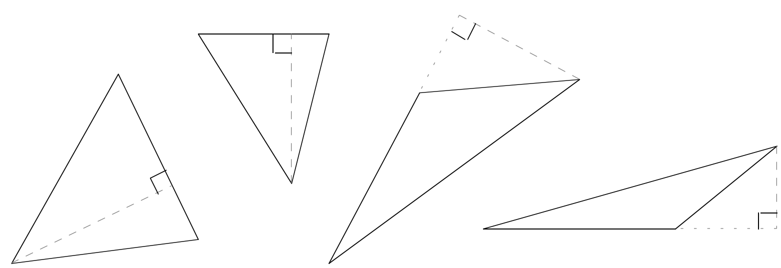
c. An equilateral triangle based on your segment

d. A square based on your segment

22. Starting with an equilateral triangle, construct a circle that passes through all three of the triangle's vertices.

23. Illustrate each of the following definitions with a sketch. The first is done for you as an example.

a. A triangle has three *altitudes*, one from each vertex. An altitude is a perpendicular segment from a vertex to the line containing the opposite side.



To keep the pictures understandable, only one altitude is shown for each triangle. You may wish to trace each picture and draw in the other two altitudes.

b. A triangle has three *medians*. A median is the segment connecting a vertex to the midpoint of the opposite side.

c. A triangle has three *midlines*. A midline connects the midpoints of two sides.

24. Draw four triangles. Use one for each construction.

a. Construct the three medians of the first triangle.

b. Construct the three midlines of the second triangle.

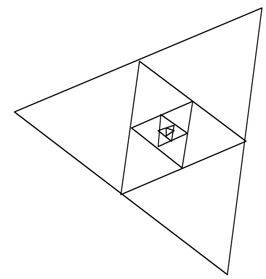
c. Construct the three angle bisectors of the third triangle.

d. **Challenge** Construct the three altitudes of the fourth triangle.

25. Compare your solutions to Problem 24 to other students' solutions. What are the similarities and differences in your results? Write any conjectures you have.

26. Start with a square. Then construct its diagonals. Study the diagonals—how their lengths compare, what angles they make with each other, how they divide the square's area, and so on. Record your observations.

27. Construct a picture of triangles within triangles like this:



Each smaller triangle is formed by the midlines of its "parent" triangle.

CHECKPOINT

28. In this section, you have run across many more mathematical terms that have special meanings. Look back through your work and list the ones you've used in your writing.

29. Look back through the reading and problems in this section and list any terms about which you are still uncertain.

30. Write what you know about each of these terms. Explain what they mean as precisely as you can.

line	invariant
perpendicular	line segment
vertex	tangent
altitude	bisector
midline	median
diagonal	

31. Can you make a triangle with the following angles? (Try to answer without constructing the triangle.)

a. $50^\circ, 50^\circ, 50^\circ$

b. $60^\circ, 60^\circ, 60^\circ$

c. $45^\circ, 45^\circ, 90^\circ$

d. $72^\circ, 72^\circ, 36^\circ$

e. Can you make a triangle with two 90° angles?

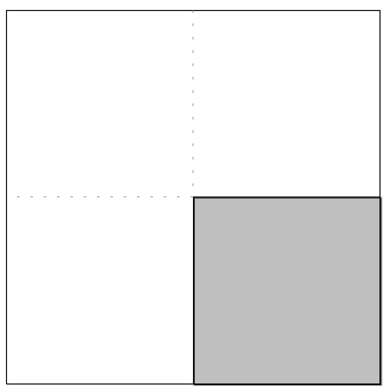
TAKE IT FURTHER

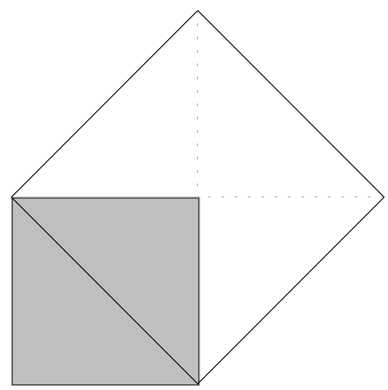
32. Construct a square. Now, without measuring lengths or areas, construct a second square that you *know* has exactly twice the area of your first square. How did you do it?

PERSPECTIVE ON IMPOSSIBLE CONSTRUCTIONS

Geometers over the centuries have been fascinated with the challenge of constructing figures using only the two basic tools: a compass and a straightedge. This essay introduces three famous impossible constructions and the history surrounding them.

Problem 32 may be difficult, but it can be done. Well over 2000 years ago, scholars knew that if you doubled the length of the sides of a square, the area would be quadrupled, but that if you built a square on the diagonal, the area would be doubled.





There were other constructions, however, that the ancient mathematicians were never able to do with just a compass and straightedge as tools. Three famous constructions—doubling the cube, squaring the circle, and trisecting an angle—have since been *proved* impossible to do if one is allowed to use *only* a compass and straightedge. Two of the constructions were not proved impossible until the 1800s. There is no evidence that the algebraic techniques used in the proofs were known by the ancient mathematicians.

Many people, when they hear about the "impossible constructions," take them as a challenge, and set out to do the constructions. But there is a big difference between something that has been proved impossible and something that simply hasn't been

solved. The proofs show what kinds of line segments *can* be constructed with a straightedge and compass, and, for example, one representing π (which would be needed for squaring a circle) is not a possible segment.

DOUBLING THE CUBE

There are two stories about the origin of the problem of doubling a cube. In one story, the Greek King Minos had a cubical tomb constructed for his son. When it was complete, he thought the tomb was too small, so he ordered the builders to double its size by doubling the length of each edge.

33. Was the king right or wrong about how to double the size of the tomb? If each edge of a cube were doubled, how would the final volume compare to the original volume?

34. A new cube is built so that each edge matches the diagonal of a face of the original cube. How does the new volume compare to the original? Is it doubled? More than doubled?

35. Suppose you have a cube with side length 1 inch. What length segment would you need to construct to make a cube that has double the volume?

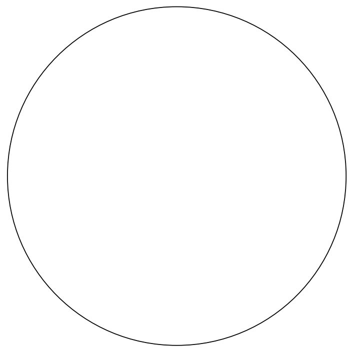
A second story about doubling a cube comes from ancient Greece: The gods sent a plague on Athens because they were unhappy with a cubical altar to Apollo. If the Greeks doubled the size of the altar, they were told, then the plague would stop; otherwise, it would continue. First they tried to build another cubical altar with each edge twice the length of the original edge, but that wasn't what the gods had asked for, so the plague continued. Of course, if the myth were true, the plague would never have stopped, since there was no way that the Greeks could have constructed the appropriate length!

Well, maybe they approximated and the plague approximately stopped!

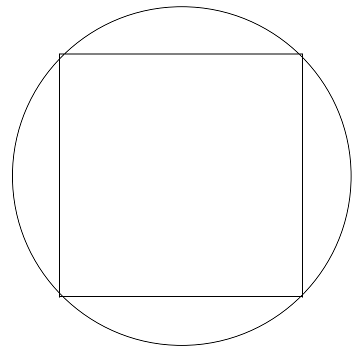
SQUARING THE CIRCLE

The problem of "squaring a circle"—constructing a square with the same area as a given circle—is very old. The Greeks knew about it by 400 B.C., and similar problems had been raised in ancient Egypt. If a circle had a one-unit radius, its area would be π units. (Area of a circle $= \pi r^2$.) A square of the same area would need to have a side of length $\sqrt{\pi}$. That length has been shown impossible to construct with a compass and straightedge, given the unit length.

Remarkably, this construction sometimes looks like it is *not* impossible. One student suggested the following construction for squaring the circle:

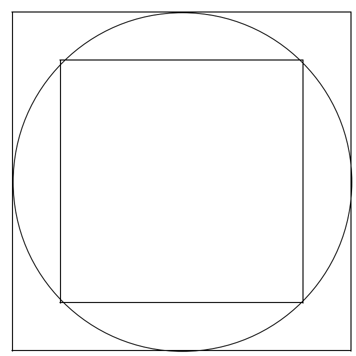


Start with your circle.



Then construct a square inside the circle, with all of its vertices on the circle.

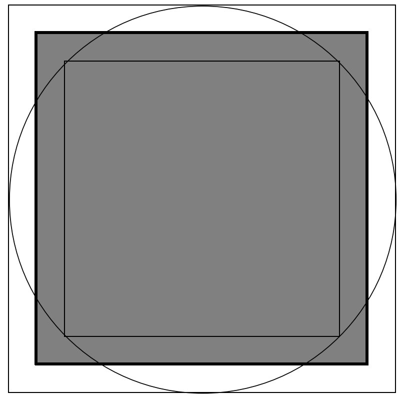
Now construct a square outside the circle. Each side is tangent to the circle and parallel to a side of the first square.



Now the first square is clearly smaller than the circle because it fits inside the circle. The second square is larger than the circle because the circle fits inside of it.

. . . or $\frac{3}{5}$ of the way or $\frac{5}{8}$ of the way or some other easily constructed distance.

But maybe the square halfway between them has the same area as the circle.



36. How could you construct the square that's "halfway between," as shown in the picture?

37. This is a good problem to investigate with geometry software, if you are familiar with using it, because the software can calculate the area ratio for you. If the ratio is 1, the areas are equal (or so close that the computer can't hold enough digits to tell the difference). If the ratio is not 1, but is invariant as you change the circle, then there's some other relationship. Use geometry software to build and investigate this conjecture. Did the student "square the circle"?

38. How are the square and the circle related in area? Geometry software can give you a decimal approximation for the ratio of their areas, but can you find the exact relationship? What is a formula for the square's area in terms of the circle's radius? How does it compare with the circle's area?

TRISECTING AN ANGLE

In compass and straightedge constructions, any angle can be bisected easily. Also, any line segment can be divided up into any number of pieces. It seems quite reasonable to believe, then, that angles can also be subdivided into any number of pieces. Well, it turns out that it's impossible (except for a few special angles) to divide an angle into *three* equal-size parts.

39. **Write and Reflect** Many people have claimed that they've found ways to trisect arbitrary angles, but flaws have always been found with their constructions.

Either they work only for special angles or they require more than just a straightedge and compass. Find out about one of these constructions and write about how it's done in your own words. Include pictures!

PERSPECTIVE BY MARION WALTER, A MATHEMATICAL MEMOIR

This section has been shortened and adapted from "A Mathematical Memoir," originally published in *Mathematics Teaching #117*, December 1986, a journal of the Association of Teachers of Mathematics, Derby, England. Printed with permission from the ATM.

Professor Walter earned her master's degree in mathematics at New York University and her doctorate in mathematics education at Harvard University. She is internationally known in the field, perhaps especially for her work in informal geometry and in helping students become mathematically creative—helping them learn how to invent and investigate new problems. She has taught extensively and has written numerous books—for children as well as adults—and many more articles, that combine her life-long interests in mathematics, creativity, art, and design. In this essay, she tells about some of the childhood experiences that moved her towards mathematics and teaching.

My father was in the bead business and I had boxes full of beads to play with. I must have done much threading, counting, and pattern making. Could my interest in symmetry stem from trying to make "balanced" necklaces? I also had sets of parquet blocks and I loved to make the designs provided on sheets. Is that how I got my intuitive feelings for geometry?

What does Dr. Walter mean by "the missing 2"?

How do *you* do such a problem?

Of course, there were also classroom activities that influenced me. We learned with an abacus (10 rows of 10 beads each). I am pretty sure of the importance to me of learning with an abacus as I never had to "learn" my number facts—I just knew them from my experience with the abacus. Numbers that added to 10 were very easy, as were ones that added to less than 10. Then sums like $8+7$ were easy; even today, as soon as someone says 8, I *feel* the missing 2, and the 7 obliges by "breaking up" into $2+5$. I often wonder why research is not done on *how* people cope with numbers and with what model they learned. I often ask my students to add two numbers, such as $27+15$, in their head and then to tell how they did it. Some are "breaker-uppers" but many are column-adders. I am definitely a breaker-upper and I am sure it is because I learned with an abacus. I might add $27+15$ by thinking $20+7+15 = 35+7 = 42$ or $27+15 = 30+15-3$ or $27+15 = 27+10+5$. I couldn't possibly do it as the column-adders do it: "$5+7 = 12$, so write down a 2 and carry the 1," and so on. My memory is far too poor to deal in my head with "carrying" and recalling digits.

After attending a public school, I went to a private, very progressive, Jewish boarding school which had strong emphasis on arts and crafts and the outdoors. I remember only a few things about academic school work there. One of the things I do recall is saying our times tables forwards and backwards; we made a game of it 7, 14, 21, 28, 35, . . . , 70, 63, 56 . . . . We did not say $1 \times 7 = 7$, $2 \times 7 = 14$, ... I seem to recall it was obvious that 35 was 5×7 and not 4×7 or 6×7. Because I *talk* very fast, I was the quickest in saying these tables. It was during this time that I must have begun to believe that mathematics was safe. $7 + 17 = 24$, no matter what, and it was safe to say it.

Reading this history, why do you suppose Dr. Walter especially appreciated, as a child, that there were certain things that were safe to say? She emphasizes this idea later in the essay, too.

My sister and I arrived in England on 16 March 1939, and at an English boarding school in Eastbourne the very next day. Not a word of English did I know. Knowing that we spoke no English, the headmistress immediately put us into a mental-arithmetic class, since numerals are language-free. Dozens of problems were written in a booklet and one put up one's hand when a problem was completed. My sister and I always finished before anyone else! A few days later during regular classes, I found myself in what must have been beginning algebra. Here, not speaking English may have been an advantage! Since I could not understand what the teacher was explaining, I had to find my own reasons for things. I found solving linear equations no problem at all—perhaps we should more often let students figure things out for themselves.

We had geometry along with algebra and arithmetic, and did much work with straightedge and compass. When I heard, in class, that one could not trisect an angle with straightedge and compass, I simply didn't believe it! My teacher diligently checked my many—and often very long—"proofs" of how one *could* trisect any arbitrary angle, and pointed out my errors. It was not until college that I believed that the task was truly impossible. That may have been a good thing, for I learned a lot of geometry trying to show how one *could* trisect angles.

School Certificate is a graduation exam. If you pass, you graduate.

In 1944, at the standard age of 16, I took School Certificate. Because I got a distinction in mathematics, the headmistress called me during the holidays to tell me that the mathematics teacher had resigned and that there was no way that the school could get a new one because of the war shortage. Since she knew I had not decided what to do, she asked me if I would come back and teach mathematics! I went back for two terms to teach in upper school. I still had to sleep in the student dormitory, but I was now allowed in the staff room, had an afternoon off, and was paid a small amount. In the summer, the headmistress asked me if I wanted to teach. If I did, I would have to go to college.

Because I had not been planning to go to college, I hadn't spent my school days worrying about passing tests to get in. I feel sorry now for children who do worry so

much about things like that. Luckily, the headmaster took a chance on me. He knew that I had no physics or chemistry in school. Though those courses were a prerequisite, he accepted me provided I studied some over the summer. (He did not know that I was afraid to light a match—which made my physics and chemistry classes a bit difficult!) In later years, I was able to visit him and thank him for taking a chance. I did not find college easy, but I took and passed the Intermediate Bachelor of Science degree before moving to New York. Just crossing the Atlantic made me a better student! I was amazed to find that in America one took 15-week courses and was examined on each course separately, and directly afterwards. This made exam-taking much easier, though learning less thorough. I spent several hours each day tutoring high school students. After two more years, I obtained a degree from Hunter College in New York and a teaching license.

I have been influenced by many great teachers including the mathematician and teacher of heuristics, George Polya, with whom I was fortunate to have a course during a summer institute at Stanford. One of the most valuable courses for *teaching* mathematics I ever took was a three-dimensional design course I had at Harvard University. In a sense it was the most thinking course I ever took. After all, in mathematics, and certainly in science, one is given some techniques and one has ways of checking if one is right. In the design course, we had to find out on our own the quality of paper, wood, and so on; and we had to *find* ways to check the validity of our designs—no answer to look up, no substituting a value to check. The practical experience in this design course greatly influenced my work in informal geometry throughout my career.

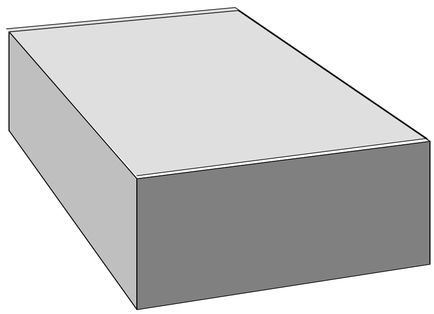
Of course, everything requires *some* memory. So, what might Dr. Walter mean by "mathematics requires no memory"?

It was partly because I felt mathematics requires *no* memory that I liked it best. Of course, there was also the feeling that it was safe—one could not get into trouble for making a statement about mathematics. Beside my strong interest in mathematics and teaching, I've always felt a strong connection between mathematics and art. (Lately, I have been teaching a college course on "Links Between Mathematics and the Visual Arts.") I still like to work on mathematics, and most of the time it gives me pleasure. I often feel that the pleasure a pleasing piece of art gives—and the pleasure of indulging in some art—is the same as is given by a pleasing piece of mathematics or by working on a piece of mathematics.

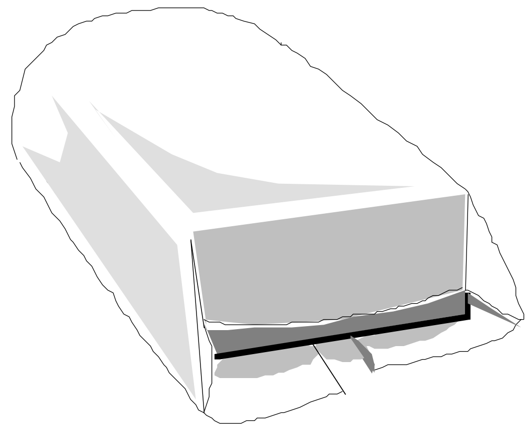
Investigation 1.6

Constructing From Features: Paper Folding

Wrapping a birthday present often involves folding a flat piece of paper into a 3D right-angle corner. There's some neat geometry here. If you want the wrapping to be neat, too, then it is useful to understand the geometry!



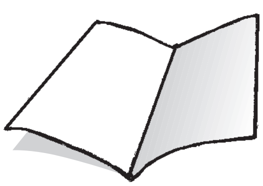
An unwrapped box



Partially wrapped

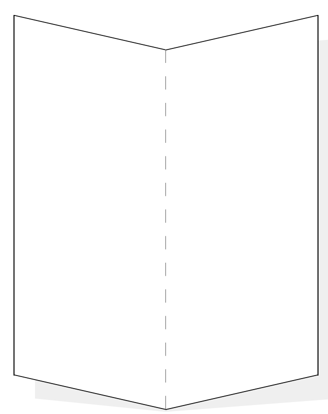
In the second picture, the paper is not quite long enough to go all around the box, so there's a gap at the bottom. When the top is bent down to meet the box and the paper is creased neatly so that it meets the side edges of the box, the crease makes what looks like a 45° angle. In fact, regardless of the size of the paper or the measurements of the box, it *does* make a 45° angle!

The following construction creates a right-angle corner without using a box to guide the folding. Try it, and then think through the questions at the end that ask you to explain why the 45° angle makes the three planes perpendicular to each other.



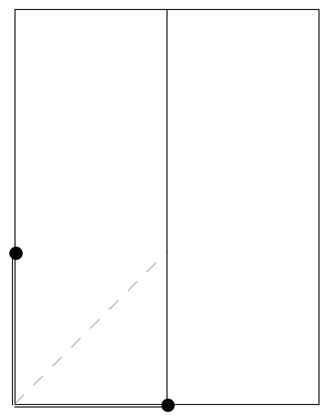
A valley fold

Step 1. Fold a rectangular sheet of paper the long way. (The dashed line indicates a "valley fold"—a fold that remains low while the paper around it is raised higher.)

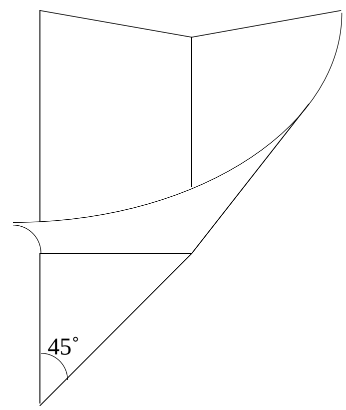


Fold the long way.

Step 2. Making the 45° angle: Bisect one of the lower angles. Crease only to the center fold, and then unfold the paper again.



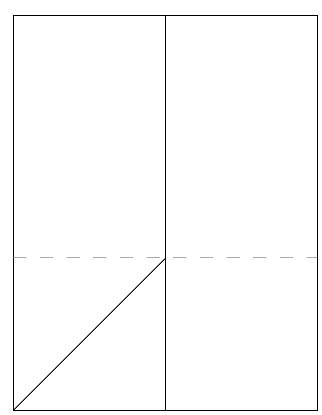
This is the crease you need to make.



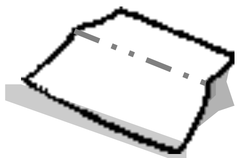
While you fold, it looks like this.

The "crease pattern" for an origami creation is the pattern you see in the paper after the creation has been unfolded.

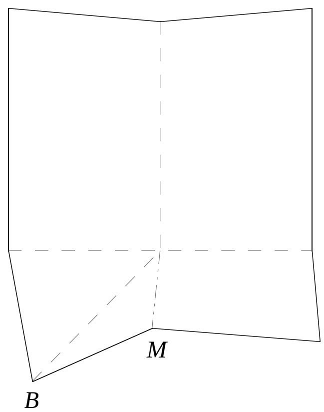
Step 3. Fold the bottom edge upward making a crease through the point where your two other creases meet. Then unfold again so that the crease pattern looks like this.



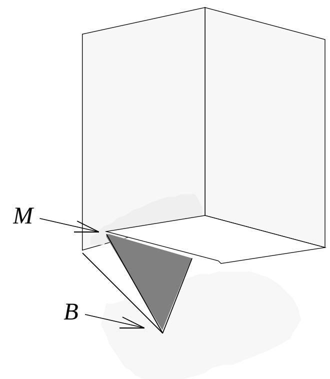
The dash-dot-dot pattern on a crease indicates a "mountain fold"—a fold that is higher than the surrounding paper.



Step 4. Now, begin to gather the paper together as these pictures show.



Midpoint M folds up; corner B tucks under.



When finished, the result looks like two walls and a floor.

You now should have a right-angle corner, three planes that are all perpendicular to each other.

1. In fact, there are several right angles in your construction. Draw a picture of a corner, label important parts, and mark all of the right angles in the corner.

2. Repeat the construction process with a new piece of paper, but make a different angle in Step 2. (Don't make a 45° angle.) Explain what happens. (Draw a picture!) Are the three planes perpendicular? Is there a pair of perpendicular planes?

PERSPECTIVE ON ORIGAMI MATHEMATICIANS

The ancient art of origami lives on today in the work of artists and mathematicians. In this essay, you will read about

Origami, the art of paperfolding, began many centuries ago in both China and Japan. Traditional models are folded from a single piece of paper (usually a square) without any cutting, pasting, or decorating. Perhaps the best known classic designs are birds—the paper crane, and a bird with flapping wings—but this long tradition has left a rich legacy that also includes other animals, plants, and boxes.

two contemporary origami mathematicians and their work.

Recent innovations in folding techniques have greatly expanded the possibilities. By studying the geometry behind the folds, today's origami designers create models with incredible realism and detail: a zebra with stripes, a cuckoo clock, a model of the Statue of Liberty—all from single sheets of paper! There are many good books that can introduce you to this ancient art.

Tom Hull, a mathematician fascinated with this art, writes:

"Many people ask me how I became interested in origami, the art of folding paper into animals, people, and objects. I started when I was eight years old, due to the influence of my mysterious Uncle Paul. I only met him once in my life, on a family trip. He was something of a hermit and had, in his solitude, been folding paper for a number of years. His origami birds and animals fascinated me, and as a parting gift he gave me my first origami book. On the car ride home I tried folding my first model and was instantly hooked. Little did I know that my Uncle Paul would remain so mysterious, for my family has lost all contact with him. He probably never suspected that in me he had planted a 17-year origami habit!

"My interest in mathematics is almost as old as my fascination with origami, but both stem from the same foundation—the pleasure I find in studying pattern. By this I mean not only physical patterns found, say, in the creases made by an origami model, but also structural patterns—patterns that lead to theorems that describe the laws governing how origami works. This thirst to know *how it all works* is the driving force behind much of my origami and mathematical interests.

"This has led me down a very exciting path which has melded my origami and mathematical worlds. I have joined the efforts of a handful of mathematicians around the world who are trying to study the limits of origami mathematically. In origami, powerfully complex rules are at work. Perhaps these rules can be explained and understood via some branch of mathematics. Or perhaps these origami rules are complicated enough to tell us something completely new about mathematical structures!

"I see it as my personal job not only to do more research on the origami-math connection, but to tell the greater mathematical community of the wonderful work that has already been done. Mathematicians really like this stuff, and I get more and more university invitations to speak on the subject. I hope to keep up the momentum by writing more articles, composing a comprehensive book on *Origametry*, and speaking up!"

Rona Gurkewitz, a professor of Mathematics and Computer Science at Connecticut State University and another "origami mathematician," writes:

"I have been interested in math and origami for a long time. There are many opportunities for using math in origami since origami is an open field of engineering with paper.

"I have been interested in constructing polyhedra out of multiple pieces of paper that 'lock' together without glue. This style of origami is called modular or unit origami. Along with a co-author, I wrote a book of such designs. I wrote this book hoping it would be used in math classrooms.

"This book is different from others in that we define 'systems' of origami polyhedra. The polyhedra in a system are related because they are constructed from the same or closely-related unit modules. (Two modules are related if they start with a different shape paper, but use the same folding *algorithm*. That is, the same sequence of folds is done, but with a different starting shape.)

"Interesting questions to ask are: How can I generalize a unit module to make a related module? How many different polyhedra can be made with a given unit module?"

MATHEMATICAL IDEAS FROM ORIGAMI PATTERNS

SYMMETRY

What does *symmetric* mean?

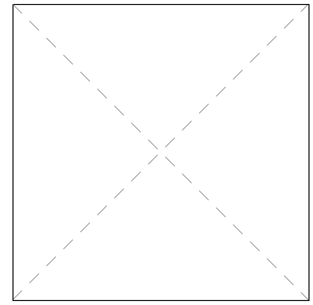
The Flapping Bird—an elegant paper bird that can actually flap its wings and "eat"—is an origami favorite. The directions below tell you how to make your own paper bird. As you work, notice how the folds often come in symmetric pairs—for example you might make the same fold on the left and right, or on the front and back. This symmetry can be seen in the bird itself, and also in the "crease pattern" (the pattern of creases in the unfolded bird), which you'll examine at the end.

Slightly different instructions for folding the same bird appear on pages 179–182 of Martin Gardner's book *The Second Scientific American Book of Mathematical Puzzles and Diversions* (New York: Simon and Schuster, 1961).

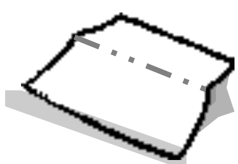
CONSTRUCTING FROM FEATURES: PAPER FOLDING (continued)

You can use origami paper, but writing or wrapping paper will do as long as it is cut perfectly square before you start the instructions below.

Step 1: Start with a square piece of paper. Valley fold along a diagonal of your square. (If your paper has color on one side, fold that side *inside* the valleys.) Open the paper and valley fold along the other diagonal.

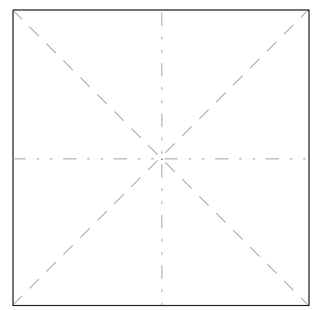


Remember: in a "valley," the fold goes down; in a "mountain," the fold goes up.

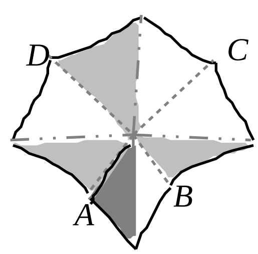


Mountain fold

Step 2: Unfold again. Now mountain fold the square in half *parallel* to the square's sides (not along the diagonal). (If the paper has color on one side, the color goes *outside* for this fold.) Unfold, and mountain fold the square in half the other way, perpendicular to the fold you'd just made.

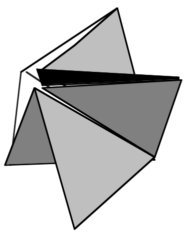


Step 3: Unfold everything. Your paper should look like this.

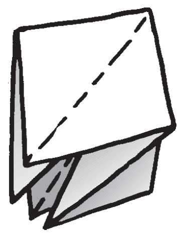


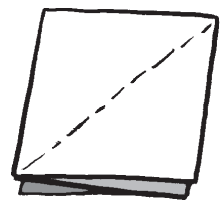
Step 4: Gather all four corners of the original square. The result should look a bit like a four-pointed star.

The four "wings."



Bring the points of the "star" together in pairs as shown below, and flatten them neatly along the existing creases. The result will be a square that is $\frac{1}{4}$ as big as your original square.

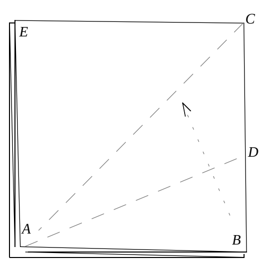


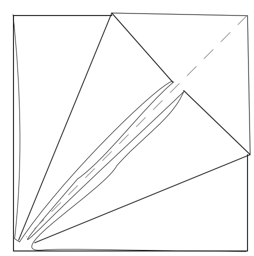


Step 5a: Two of the edges ($\overline{AB}$ and $\overline{AE}$) of this small new square are composed of edges of the original square. The other two edges ($\overline{BC}$ and $\overline{EC}$) are composed of folds.

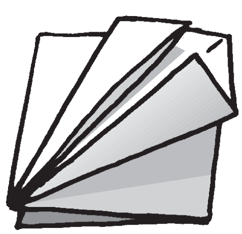
In this step, you are creating angle bisectors. Which angles are bisected?

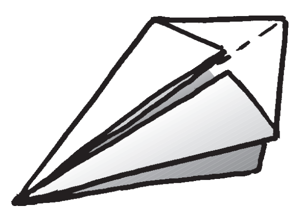
Fold edge $\overline{AB}$ so that it lies along the diagonal $\overline{AC}$. Do the same with edge $\overline{AE}$.



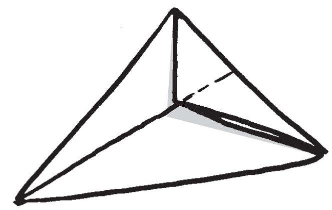


Step 5b: Turn the paper over and repeat on the other side.

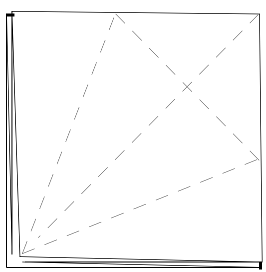


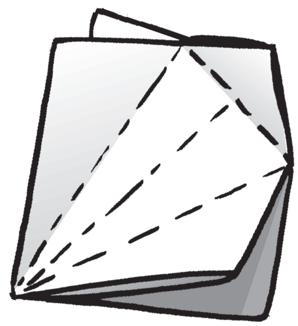


Step 6: Fold down the top of your kite and make a crease.



Step 7: Now unfold the creases you made in Steps 5 and 6, so that you are back to your small square, and your crease pattern looks like the one below.

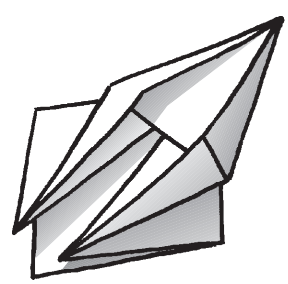




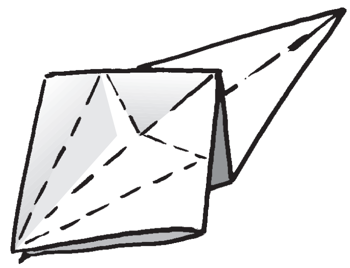
Step 8 is difficult to describe and takes careful work. Read it thoroughly and look closely at the picture before you start. You will make *no new folds* in Step 8. All you will do is "reverse" some of the creases you've already made (that is, change them from mountain folds to valley folds, or vice versa).

As you open the top layer, the two side corners will pull in, as the picture shows. *Let them pull in*, and then press them flat using only the creases that already exist, the ones you made in Steps 5 and 6. It will mean reversing some creases.

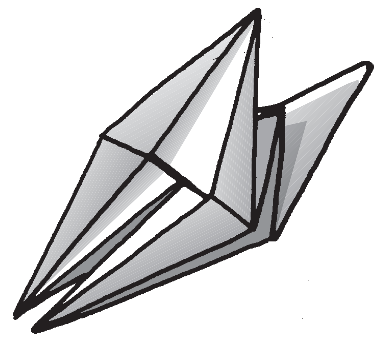
Step 8: Open up the top layers of your square, and fold back along the existing creases. When flat, the top layers will form a narrow rhombus. (The bottom layers still form a square in this step.)



Step 9: Turn the paper over and repeat Step 8 on the other side.



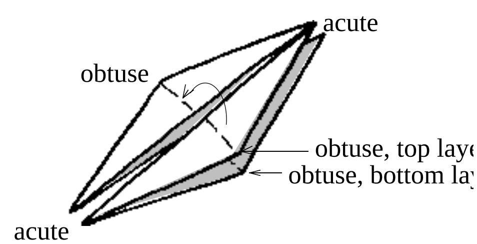
The paper turned over



Repeating Step 8 on the other side

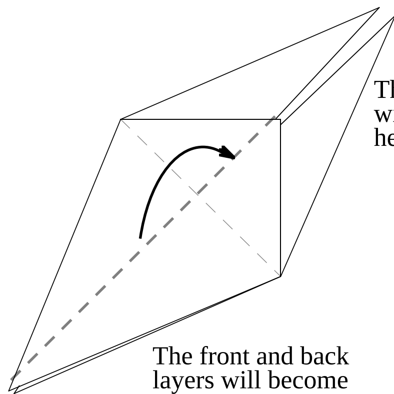
Can you figure out the measures of the two acute angles of the rhombus?

Flatten this out carefully. It will form a narrow rhombus.

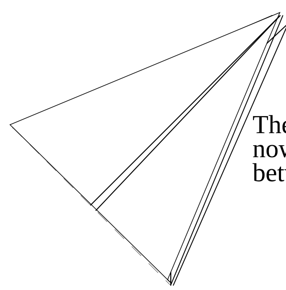


Step 10: This narrow rhombus has two acute-angle corners and two obtuse-angle corners. Raise the obtuse-angle corners of the top layer up, and then bring them together to meet each other. Then bring the corresponding bottom corners *down* to

meet each other, and flatten along the middle crease. The result will be much the same shape, like this.



Step 11: On both front and back, fold the diamond along its short diagonal, so that the triangles that will become the wings completely cover the parts that will become the head and tail.

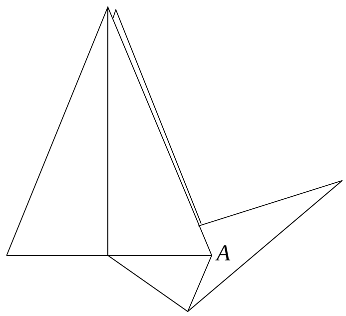


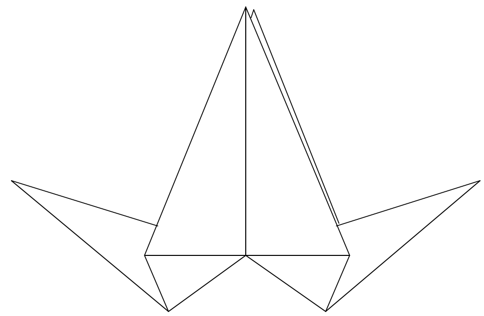
The bird, tucked tightly inside its "egg."

You can choose which is the head and which is the tail. It really doesn't matter because the folds have all been symmetric.

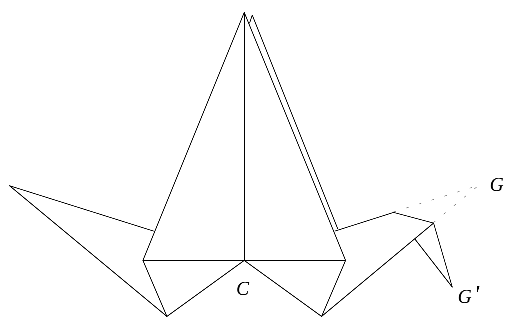
Step 12: It's time for the bird to hatch out of its egg! Pull the tail out from between the wings until you can squeeze the paper flat with point A roughly in the middle of the tail. (The exact placement doesn't really matter. Once you have made a few birds,

you can decide what works best and looks best to you.) Now do the same with the neck end.





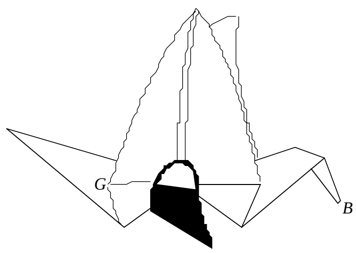
Step 13: To make the head, invert a portion of the neck by reversing the center fold from mountain to valley and tucking the head part inside the neck. The exact position or angle is not important; do what looks best to you. The bird is now complete. In Step 14, you will add the finishing touches that allow it to flap its wings gracefully and "eat."



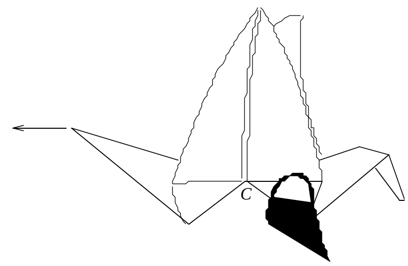
The first attempt to flap the wings of a newly hatched bird is often difficult. Don't pull too hard or the paper will rip at C. If the wings don't flap easily at first, help them a bit by gently pushing them away from the tail as you pull the tail.

Step 14: A few finishing touches. Hold the bird with your thumb and fingers over the "belly" (as in the picture), *curl (don't fold)* point *A* gently away from the tail, and curl (don't fold) the wing gently forward toward the head. (Do the same to the left wing.) Now, hold under the neck and *pull* (don't "pump") the tail *gently* away from the head. The wings will flap!

At the beak (B), tear (don't cut) the tiniest bit of paper off. Now let the beak "peck" at the tiny "seed" you just removed. It will pick it up. Can you figure out why this works?



How to hold while curling the wings



How to hold while flapping the wings

The first time through may seem quite complex, but you will probably know the directions by heart after folding only a very few more birds.

3. Unfold your bird, so you can look at the whole crease pattern. Draw a picture of the crease pattern.

For this problem, you can ignore the small crease made by bending the head down. Can you identify which crease that is?

4. A line of symmetry on your crease pattern is a line you can fold along so that every crease on one half matches with a crease on the other half. How many lines of symmetry can you find? Mark these lines in a different color or make them darker in your drawing from Problem 3.

Tom Hull says: "A mathematician first showed me this puzzle and claimed that five folds was the fewest. But being an origamist, I was able to do it in four!"

5. **Constructing Lengths** Starting with a square piece of paper, fold it into a square with $\frac{3}{4}$ the area of the original square, and do this in as few folds as you can.

WAYS TO THINK ABOUT IT

If your original square of paper has area 1, then the square you want to construct will have area $\frac{3}{4}$. A square having a particular area has side-

sidelengths that are the square root of that area. The challenge is to construct a new length that is

$$\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{\sqrt{4}} = \frac{\sqrt{3}}{2}$$

unit long given a square with sides that are 1 unit long.

You know that you can always divide a length in half by folding, so the question is really how to create a length of $\sqrt{3}$ units through paper folding alone.

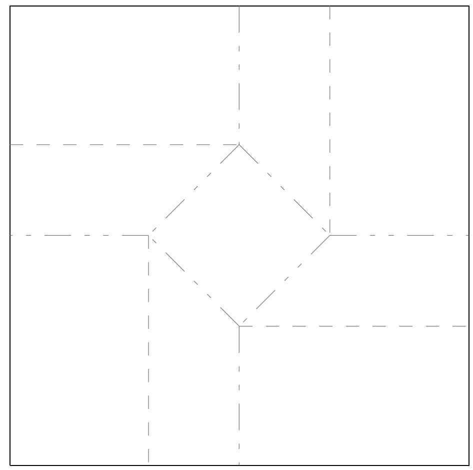
A more general question: What lengths *are* constructable given a unit length and particular set of tools? Using ruler and compass alone, certain lengths cannot be constructed, and that is why, for example, those tools alone cannot be used to construct a square that has the same area as a given circle.

THREE-DIMENSIONALITY AND TWO-DIMENSIONALITY

To wrap a box, one must crease a plane so that it will fold into a three-dimensional shape. Many paper foldings, like the one in Problem 5, lie flat (perhaps several layers of paper thick, but not deliberately three-dimensional). A question that interests several origami mathematicians is how to tell, just by looking at the crease pattern, whether the pattern can be folded at all and, if so, whether it will fold flat or will remain three-dimensional.

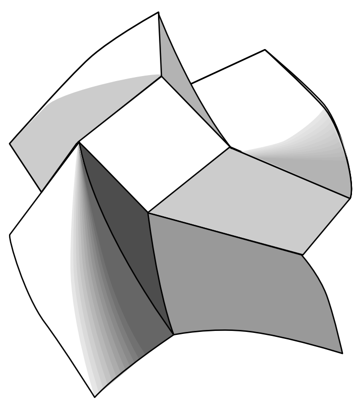
Even though there are very few creases, this takes patience and care to fold. However, when you succeed, you will see why the fold is called a square twist.

6. Trace the crease pattern on the next page onto a square piece of paper. Create the mountain and valley creases carefully. Then fold along the creases. With patience, you will get a special type of origami fold called a *square twist*.

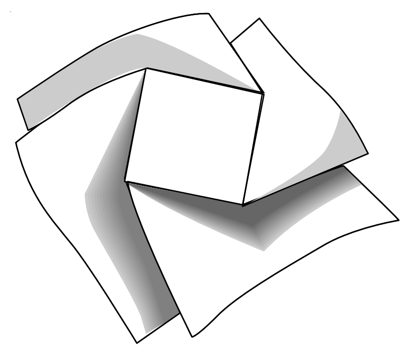


The dashed lines are valley folds, and the dash-dot-dot lines are mountain folds.

The figures below show one way to fold up a square twist.



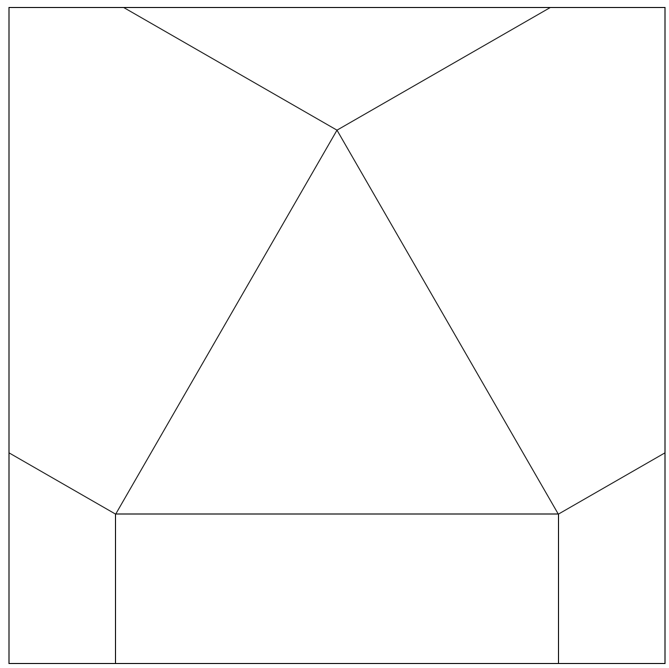
Partially folded



Mostly folded

7. Is this a fold-flat origami or a three-dimensional structure?

8. Here is a crease pattern that may seem very similar to the square twist. You might expect it to produce a "triangle twist," but, in fact, it cannot be folded at all unless you add more creases.



Decide where to place an additional three creases so that you can fold this into a three-dimensional structure that resembles a tablecloth draped over a triangular table.

Investigation 1.7

CONSTRUCTING FROM FEATURES: GROUP THINKING

Each member of your group will have one (or at most two) clues about a drawing.

Do not show your clue to anyone else. You may read your clue aloud, draw a sketch based on your clue, or discuss the clues and drawings, but you cannot *show* your clue.

Together, your group must make and agree on a drawing that fits *all* the clues that the group has. Sometimes, more than one drawing will fit the clues. Each person must then be prepared to explain why your group produced each of its figures.

CHECKPOINT

Your teacher will give you a card that describes a geometric term or construction. Some of the constructions are easy; a few of them may be impossible without the correct tools.

1. Make the construction on your card, explaining *how* you created it, especially if you come up with a clever method.

2. Figure out how to make the construction with *only compass and straightedge*. Some may be impossible with only those tools. If this seems to be the case, think about what other tools are needed. Does a ruler make it possible? For which constructions do you need a protractor?

3. Now that you know how to make these objects, write directions for someone who may not know the definitions of many of these geometric terms (for example, a fifth grade math student).

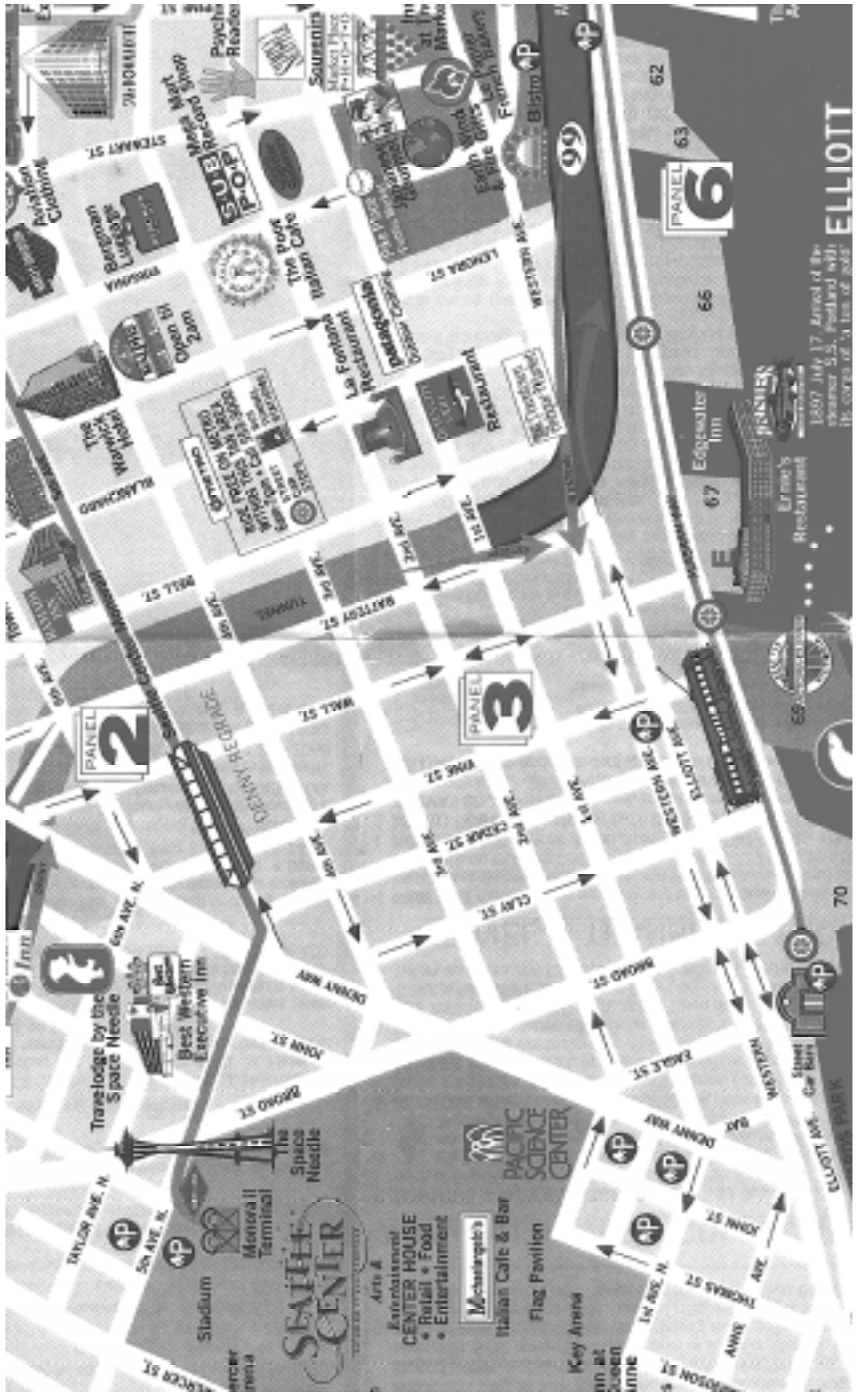
A restriction: Any word that is part of the name of an object cannot be used in your instructions for drawing that object. For instance, the word *perpendicular* cannot be used to describe how to make a perpendicular bisector.

TAKE IT FURTHER

4. Write a set of clues, like those from this investigation, that a group of students can use to construct some figure.

Investigation 1.8

ALGORITHMIC THINKING: DIRECTIONS FOR PEOPLE



When you give directions to help a person get from one place to another, you are describing a path in space. When you follow such directions, or draw a map from them, you are interpreting a description of a geometric process.

When a process is completely specified (no need for intuition or judgment, and no need to add "well, you know what I mean") and when the process's outcome is fully reliable, it is called an *algorithm*.

The word *algorithm* is a distorted transliteration of a great Islamic mathematician's name. abu-Ja'far Muhammad ibn Musa was known as al-Khwarizmi, which means "the one from Khwarazm." The Latin attempt at spelling al-Khwarizmi was Algorismi, which later became algorism and then algorithm.

Algorithmic thinking—being able to create, interpret, and reason about processes or procedures—has always been a very important part of mathematics, and its importance has become even more widespread since the advent of computers.

In this activity, you will describe and interpret geometric processes by writing directions in "natural language" for people to follow. In the next activity, you'll write similar algorithms using a formal computer language.

Here is a set of directions from the Pike Place Market to another famous destination in Seattle:

Directions from Pike Place Market
Come out of Market. Turn L onto 1st Ave. Go about 11 blocks. Sharp R onto Denny Way. Go a few blocks, until Broad. L onto Broad. Pass John St., and turn into parking lot, left.

1. a. Draw a map that could go with these directions.

b. Compare your map to someone else's to see if the line segments and angles match up. Describe how the maps are similar. How are they different?

Pike Place is labeled "Public Market"—it's near the clock face just above "Japanese Gourmet," toward the right-hand side of the tourist map.

2. Look at the tourist map of Seattle. If you started at Pike Place Market and followed the directions given, where would you end up?

3. The given route is not the shortest one between the two sites. Find a shorter route (by car only), and write directions for it.

FOR DISCUSSION

Explain why you agree or disagree with this statement: "For someone who is actually following the path on the streets of Seattle or who is marking a path on a map of the city, the directions given on the previous page are an algorithm. For someone *drawing* a map of Seattle, the directions are not an algorithm."

CHECKPOINT

If the layout of your school is complicated, it can take new students a long time to learn their way around.

4. Write directions for travel between two locations in your school. For example, how would you tell a new student to get from your mathematics classroom to a rest room? Be clear but brief; give no unnecessary information.

Drawing the map can be messy if stairs are involved.

5. Draw a map that goes with your directions for Problem 4.

6. Borrow a classmate's written description and draw a map of the path described. Compare the maps that you and your classmate drew from the same written description. In what ways are they the same? In what ways are they different? What knowledge did you need to have, beyond what was spelled out in writing, to draw the path correctly?

7. Write a set of directions from school to your home. Make them good enough to guarantee that someone who will walk or drive that route will reach your home.

Investigation 1.9

Algorithmic Thinking: Directions for Robots

Computer Tools

Instead of writing directions for a person, you will now type directions to a robot. The kind of robot you'll use is called a turtle. Describing a path to the turtle is like telling it how to *draw* the path. It must be told both what *kind* of movement to make (go forward, go back, turn right, turn left), and *how big* that movement is to be (how far to go or how much to turn). A little later, you will use a second computer tool called geometry software.

The Logo turtle is a movable shape on a computer screen. The early turtles were robots designed to be used on the floor. The first one was shaped a bit like a small upside-down yellow trash can—with a hard shell on top and wheels on the bottom—and was a large and heavy thing that moved slowly about on the floor like a . . . well, like a turtle. In memory of these early lumbering robots, objects on the computer screen that have the same *behavior* are generally called "turtles," too.

Turtle Geometry

A computer "turtle" can be given instructions to rotate specific angles or proceed forward specific distances. By writing sets of instructions for the turtle to follow, you can create precise drawings. The instructions are written in a programming language, often Logo.

Geometry Software

A bit like computer versions of hand construction tools, these programs allow you to construct figures using specific features. They also let you experiment with your construction—bringing it to life by animating its parts—so that you can watch for invariants *as you make certain changes*.

A robot must also be given directions in a language it understands. This particular robot understands a language called Logo. Here is a version of the directions from Pike Place Market to the Space Needle, adapted from the Seattle map and written in Logo, with a translation into English.

Turtle-Talk (Logo)	*Person-Talk (English)*
To PikeToNeedle	*(a title for the algorithm)*
left 90	*Turn L onto 1st Ave.*
forward 12.5	*Drive on 1st Ave.*
left 15	*1st Ave. bends to the left.*
forward 100	*Drive the rest of the 11 blocks.*
right 135	*Take a sharp right onto Denny Way.*
forward 25	*Go a few blocks.*
left 45	*Turn left onto Broad St.*
forward 13	*Pass John St.*
left 90	*The parking lot is on your left.*
forward 5	*Drive into the parking lot.*
end	

Your Logo may provide a special place or "window" for typing procedures. If you aren't sure where this place is, ask.

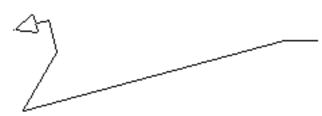
To clear the graphics screen between experiments, use either CS for "clear screen" or CG for "clear graphics" (depending on the Logo you use). Always follow a command with *RETURN*.

Do you know how to save the procedures you create or how to print the pictures you have drawn? If not, ask.

1. Create a procedure called PiketoNeedle. Type it *exactly* as shown.

Test the procedure. To see what the robot does with these instructions, type: PiketoNeedle at the robot turtle's command center, then press RETURN.

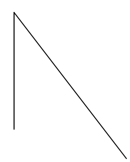
Your procedure should draw a picture like this one. If it doesn't, check to see if you have typed all instructions exactly as given.



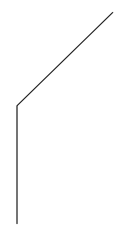
2. Print a copy of your path or make a sketch of what's on the screen. Label all parts of your picture to agree with the labeling on the Seattle map. Why is Denny Way "at an angle"? What *is* the angle between Denny Way and 1st Avenue?

FOR DISCUSSION

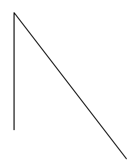
Look at the following pictures and think about what you mean by “angle” when you are drawing, when you are riding a bicycle or driving, and when you are measuring. Think of everyday situations to go with the following pictures.



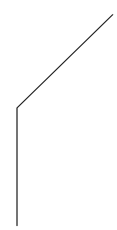
A small angle



A large angle

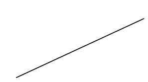


A large turn



A small turn

No angle



At an angle

Remember that the robot turtle has no judgment at all. The turtle follows its directions precisely and literally. If you want to check a set of directions by "playing turtle," you must refuse to do anything that isn't exactly specified by a distance or angle.

3. Write a procedure called Block for going around a rectangular block (arriving back where you started and facing your original direction). The procedure will start like this:

```
To Block
 forward ...              You must fill in a distance here...
     .                    ...and the rest of the procedure.
     .
     .
end
```

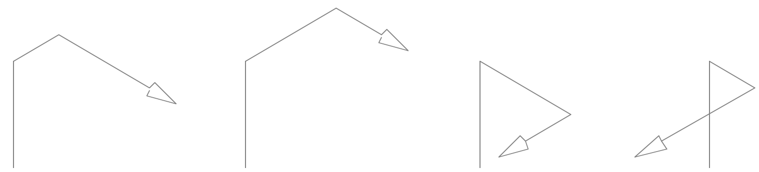
Try out your procedure after it is defined by typing Block and pressing RETURN.

CHECKPOINT

Do this problem *in your head,* without testing the procedures out on the computer.

4. Match each procedure to the picture it draws on the next page.

```
To Picture1          To Picture2
  forward 40           forward 40
  right 120            right 60
  forward 40           forward 40
  right 120            right 60
  forward 20           forward 20
end                  end

To Picture3          To Picture4
  forward 40           forward 40
  right 120            right 60
  forward 20           forward 20
  right 120            right 60
  forward 40           forward 40
end                  end
```



a. b. c. d.

TAKE IT FURTHER

How will you indicate stairs in your map?

5. Make a map for fire emergencies. Look near the door of your classroom for instructions on how to exit the building in case of fire. Direct the turtle to draw the path that you should follow to your exit, or at least to a stairwell.

To save typing, many turtle commands have abbreviations. `right` and `forward` may be typed as `rt` and `fd`.

6. What hand construction tools would be needed to draw as the turtle draws? (For example, what would be necessary for you to follow, by hand, the commands `fd 100` or `rt 40`?) Explain which turtle movements would require which tools.

7. Suppose you wanted to have a 3D Logo turtle (as if the turtle were swimming in a pond). What new commands would you need? Explain what each command would do and why you would need it.

8. **a.** Tell the turtle to go forward without specifying how far (just type `forward` without a number). What happens? Explain the result in writing.

b. What happens if you tell the turtle to turn without specifying how many degrees?

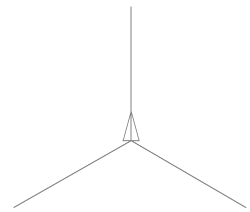
c. What happens if you provide a number without a command preceding it?

Investigation 1.10

ALGORITHMIC THINKING: ANGLES AROUND A CENTER

In this investigation, you will develop algorithms for drawing angles around a center point. By analyzing these algorithms, you will discover some properties of angles around a point.

This design is made of three lines. The algorithm for drawing it told the turtle to move out along a line and then move back along that line before turning to draw the next line.



There are many ways of getting the turtle to draw a line and then retrace its steps. Here are three.

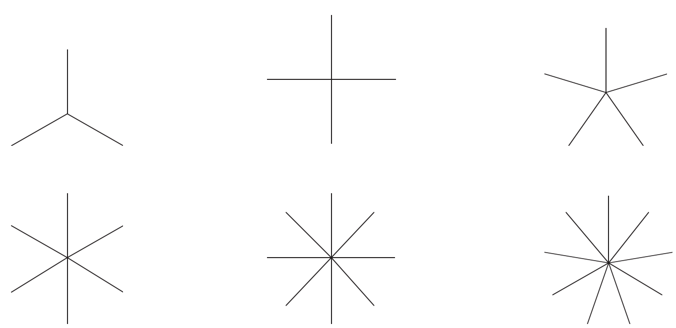
i	ii	iii
forward 25 right 180 forward 25 right 180	forward 25 forward –25	forward 25 back 25

(Why do we include "right" 180 at the end of procedure i?)

1. a. Use one of these ways to help you draw the following designs. You'll need to figure out the turn angle between spines for each design. For example, look at the first design below. How many degrees do you think are in each of the angles? How can you check that this is correct?

Keep track of the algorithms you use. A start for an algorithm for the four-spined design is given in the margin.

```
fd 25  bk 25
rt 90
fd 25  bk 25
 .
 .
 .
```



b. How can you check a procedure to see if the turtle really will end up exactly where it started, facing in the same direction? Find a method for checking, and write an explanation of your method.

2. a. Make a table showing the amount of turning required between spines for each figure in Problem 1.

Number of Spines	Amount of Turning between Spines
3	
4	90
5	
6	
8	
9	

b. Look for a relationship between the number of spines, and the turn angle between them. Write a formula or describe the relationship in words.

WAYS TO THINK ABOUT IT

Computing an invariant Problem 2b asks you to find the relationship between two numbers, the number of spines, and the turn angle between them.

One way to find a relationship between two numbers that are changing is to think about using those two numbers to compute a third number that would remain constant.

In *this* case, one number increases as the other decreases. When that happens, a sum or product may help you get a constant value.

PACKAGING INSTRUCTIONS BY CREATING PROCEDURES

Have you ever heard the expression " . . . can't see the forest for the trees"?

For now, assume that the turtle knows how to `draw.one.spine`: it will go `fd 25` then `bk 25`.

In doing Problem 1, you probably found yourself typing certain sets of instructions over and over again. Often, that kind of repetitive "low-level detail" hides more important patterns in the algorithm. It also leaves more room for error.

3. Here are two slightly messed-up algorithms that were supposed to draw one of the multi-spine shapes. In which algorithm is it easier to recognize the intended shape that is being described? In which is it easier to notice the error?

Algorithm 1	Algorithm 2
fd 25	draw.one.spine
bk 25	rt 120
rt 120	draw.one.spine
fd 35	rt 130
bk 25	draw.one.spine
rt 120	rt 120
fd 25	
bk 25	
rt 120	

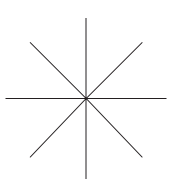
Packaging instructions in this way does more than save typing. It helps you to think of the several steps as being one thing. That can simplify *thinking* as well as typing.

Sets of instructions that can be thought of as doing a single thing—like the pair `fd 25 bk 25`, which draws a single spine—can be packaged into a single instruction like `draw.one.spine` (or, even more simply, `spine`) by creating and naming a procedure. Here's one way:

Logo	English
to Spine	*To make a spine*
forward 25	*Move forward 25 steps.*
back 25	*Move back the same amount.*
end	

To make a single spine, type `spine` (followed by a return) at the command center.

4. Type in the `spine` procedure and use it to make the spiny star shape shown here.



Reducing clutter does more than make your paper neater. It can make your thinking neater, too.

Reducing spine to one command saves typing and thinking, but the repetition in making the spiny star can be reduced even further. Eight times the turtle must do the same thing: make a spine and turn a certain amount. That way of thinking about it can be expressed in Logo this way:

```
repeat 8 [spine  rt 45]
```

Just as the command forward requires an input to tell the turtle how far to go, the command repeat requires additional information: The first input says how many times to repeat, and the second input is the package of instructions that must be repeated. (These instructions must be in square brackets.)

5. Use repeat and your spine procedure to draw each design in Problem 1.

You may remember how you've made these star designs, but the computer cannot remember (cannot "save") algorithms unless they are in the form of *named* procedures. It can save spine because that procedure has a name, but repeat 8 [spine rt 45] is a process without a name. You can give the eight-pointed star procedure any name you like:

```
To 8point                  To NicePuppy
 repeat 8 [spine rt 45]     repeat 8 [spine rt 45]
end                        end
```

The computer will run it properly regardless of the name you use, but choosing a name that suggests the procedure will help *you* recall the work you've already completed the next time you want to make an eight-pointed star.

You can try to write a 7point procedure now, but it won't be easy. (Why is that one harder?) You'll learn how to write it a little later.

6. Build a set of procedures 3point, 4point, 5point, . . . ,10point (except 7point). Save the procedures: You'll want to refer to them again in Investigation 1.11.

CHECKPOINT. . . . .

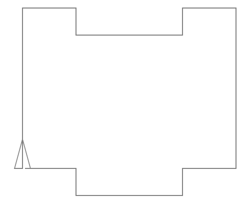
7. On the next page are two *closed* turtle paths. To create a closed path, the turtle must end up exactly where it began. Pick either one of the two paths and construct it using only the commands that are listed. Make sure that, at the end of the drawing, you leave the turtle facing *up*, as it was at the start of the draw-

ing. *Record your algorithm*. (Do not use two or more turns in a row, such as rt 90 rt 90.)

Some commands must be used more than once, of course, and you need to figure out the right order.

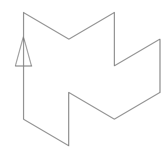
- This design can be constructed using only the following commands:

fd 10 fd 20 fd 40 fd 60 rt 90 rt −90



- This design can be constructed using only:

fd 20 rt 60 rt −60 rt 120 rt −120



8. Add up the turn angles in the algorithm you developed in Problem 7. How many total degrees did the turtle turn to get back to its original heading?

9. Write a turtle algorithm that will produce a drawing of a rectangle that is twice as long as it is wide.

10. Here is a procedure. Without running it, try to predict the picture it describes. Draw a sketch.

Include in your sketch the starting and ending positions of the turtle.

```
To MysteryFig1
 rt 45
 fd 20
 rt 90
 fd 20
 lt 90
 fd 20
 rt 90
 fd 20
 lt 45
end
```

11. Without trying them on the computer, draw the pictures that these two procedures describe.

```
To MysteryFig2
  fd 50
  repeat 4 [fd 25 rt 90]
  bk 50
end
```

```
To MysteryFig3
  fd 30
  rt 120
  fd 30
  rt 120
  fd 30
  rt 120
end
```

TAKE IT FURTHER. . . . .

12. The beginning of this investigation implied that `forward –25` and `back 25` did essentially the same thing. Without actually trying it on the computer, explain what you think the turtle would do if you typed `back –25`.

13. **Write and Reflect** Look at the following statements. Write about how they relate to multiplication of positive and negative numbers.

```
fd 25   = bk –25
fd –25 = bk 25
```

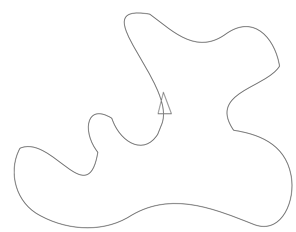
Logo can do arithmetic. For example, each of the following commands is equivalent to telling the turtle to move "forward 20."

```
fd 16 + 4
fd 2 * 10
fd 60 / 3
fd (4 * sqrt 36) – 4
```

14. You now know enough to write a procedure to draw a seven-spined figure, with the correct turn angle between the spines. Write the procedure.

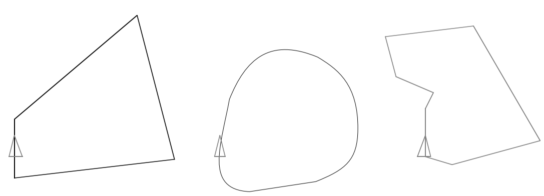
A statement of this fact is known as the *Total Turtle Turning Theorem*. (See Problem 17 for a small amendment to the theorem.)

In Problems 2 and 8, you found how much the turtle had turned from the beginning to end of various procedures that left the turtle in its starting position. The pattern in the answers was not a coincidence, and the pictures were not special cases. It can be proved that, even on wiggly trips like the one below, the Total Turtle Turning is the same amount for any closed path.



15. Come up with a reasonably convincing explanation—an informal, but reasoned argument—that the total turning in the figure above is 360°. It may help you to start with somewhat simpler figures, like the three below.

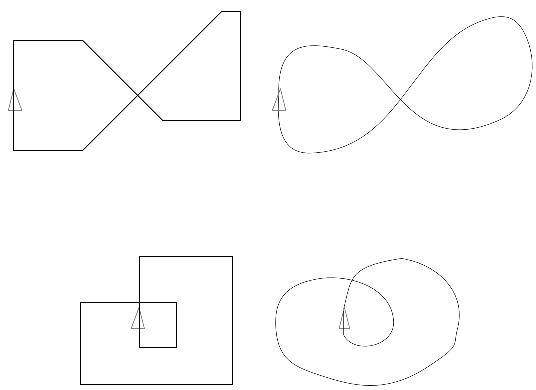
The turtle is shown in its home position—where it began and ended each trip.



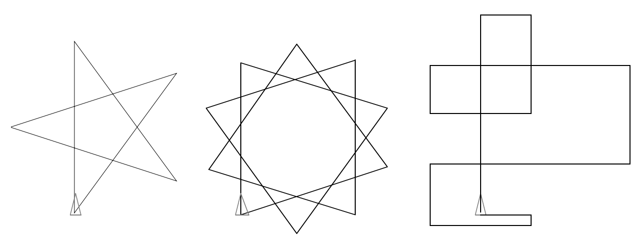
16. State the Total Turtle Turning Theorem in your own words.

17. If the turtle crosses its own path, as in the pictures below, the situation is a bit different.

a. Analyze the turning in these pictures and come up with an improved statement of the TTT Theorem. (Draw these with the turtle if that helps.)



b. What if the turtle crosses its own path more than once? Analyze these cases and try to extend the TTT Theorem to account for any number of crossings.



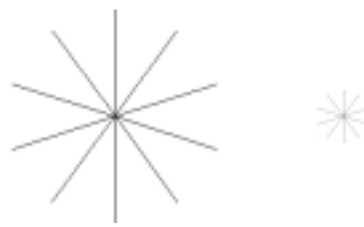
18. **Write and Reflect** Write a convincing explanation for the Total Turtle Turning Theorem.

Investigation 1.11

Algorithmic Thinking: Spines, Stars, and Polygons

The Spine procedure and the repeat command allow you to draw multi-spined figures whose spines were a fixed size. You captured that process for a ten-pointed star by naming a procedure 10point. But, the *sizes* in your procedures are all fully specified. What if you wanted to make a large star, like the one below, or a tiny star like the one beside it? It would seem *very* wasteful to create a separate procedure for each different size, when the process is really the same for all of them.

"Abstracting" and "generalizing" are central mathematical ways of thinking. Treating many similar processes as *one* process with some minor variations is a good example of this kind of thinking.



Modern programming languages allow you to define procedures with *variable* inputs. Here's an example of how it's done in Logo, showing how the spine procedure could be modified to produce variable-sized spines:

In Logo, a colon is used to mark a word or letter as a variable instead of a command. When you type the procedure into the computer, be certain to connect the colon directly with the variable name that follows it—do not leave a space between them.

```
To VarSpine :dist        varSpine is the new procedure's name.
 fd :dist                Go forward some number of steps.
 bk :dist                Go backward the same distance.
end
```

Typing VarSpine 25 as a command will do the same thing as typing Spine. But VarSpine 50 and VarSpine 10 will make larger and smaller spines. Typing VarSpine 50 sets the variable :dist equal to 50, so the turtle will substitute 50 whenever it sees :dist in the procedure.

1. Type the new procedure VarSpine into Logo. Run it with different numbers to make various-sized spiny figures.

2. Type VarSpine at the command center without giving a number; explain the meaning of the message that you get.

Using named procedures eliminates much of the repetitive typing, but still, each different spiny figure requires you to type a line that is almost the same. For each star, for example, you type:

repeat *SomeNum* [varSpine 25 rt *SomeOtherNum*]

If you don't like typing long names, you can abbreviate: just use something like :NS in place of :NumOfSpines and :TA in place of :TurnAng. The longer names can help remind *you* what the procedures do and what goes into the variables, but the computer won't care.

This common structure could be captured in a procedure:

```
To Star :NumOfSpines :TurnAng
 repeat :NumOfSpines [varSpine 25  rt :TurnAng]
end
```

Then, instead of typing

```
repeat 8 [varSpine 25 rt 45]
```

you could type

```
star 8 45
```

3. Create this Star procedure using variable names of your choice. Try it out. Make sure that it works correctly before going on to the next problems.

If you didn't do Problem 14 in Investigation 1.10, then you might not know that Logo can do arithmetic. If you type in fd 30 + 40, Logo will understand, and the turtle will go forward 70 steps.

You can do even better! In Problem 2 in Investigation 1.10, you found a relationship that allows you to figure out the correct turn angle from the number of spines. Now, because Logo can perform arithmetic, you can *use* this relationship to write a star algorithm that only needs to be told how many spines to make.

4. Create BetterStar, a procedure like Star that takes only one input—the number of spines—and figures out the turn angle from that number. Run your new procedure and make sure it works. Save the procedure on the computer so you can use it again later.

TINKERING WITH AN ALGORITHM

5. Here are two procedures that are almost identical. Compare them.

```
To Shape1 :Num
 repeat :Num [varSpine 25   rt 360 / :Num]
end

To Shape2 :Num
 repeat :Num [fd 25   rt 360 / :Num]
end
```

You be the turtle!

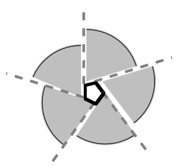
a. How are they the same? How are they different? How will the difference affect what they produce?

b. Without typing in the procedures, sketch what the following commands should produce on the screen:

`Shape1 5   Shape2 5   Shape1 10   Shape2 10`

6. Now type in `Shape2` and try it with various inputs. Sketch the results for at least three different inputs (make sure to record which input produced which shape), and describe the procedure's overall behavior. Save your work. (You might want to check your sketches from Problem 5b.)

ORGANIZING YOUR RESULTS: INTERIOR AND EXTERIOR ANGLES



The picture at the side is almost a `Shape1 5`. It is slightly exploded (taken apart) to show the angle between each spine.

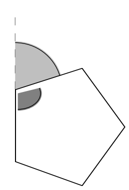
7. Write and Reflect There's quite a lot to think about in this one picture.

a. Figure out, and then explain, how this "exploded" picture *is* like the picture made by `Shape1 5`.

b. You've analyzed (in Problem 5a above) exactly how the *algorithms* for `Shape1` and `Shape2` are related. Use the picture to explain how the *shapes* produced by these two algorithms are related.

Below is a picture comparing an exterior angle of a polygon with the interior angle at the same vertex.

In the diagram, the exterior angle is pictured this way . . .



. . . and the interior angle is pictured this way:

8. Assume that the turtle used an algorithm like shape to draw such a pentagon. Which angle, exterior or interior, did the turtle use?

9. How many degrees are there in one exterior angle of a pentagon? (Use what you know about making five-pointed stars and shapes.)

10. **a.** What is the measure of the *interior* angle marked in the picture? Explain your reasoning.

b. Are all of the interior angles in the pentagon the same size?

11. **a.** What would be the measure of an *exterior* angle of regular 36-sided polygon? Explain how you found your answer.

b. What would be the measure of an *interior* angle of regular 36-sided polygon? Explain how you found your answer.

12. **Write and Reflect** If you were able to answer the questions in Problem 11, then you have a rule in your head describing how to find the exterior angles and the interior angles of regular polygons. Write this rule in words, and then translate the words into an algebraic formula.

CHECKPOINT

13. The parts of Logo that you've been using are a *formal language* for geometry. Explain the words that you've used from this language. Use parts a and b to guide your explanations.

a. Some words are "primitive" (built-in) terms. In explaining their meaning, be sure to say whether the word can be used alone or whether it needs some input (for example, a number) to work properly. If one or more inputs are needed, say what they represent. And, of course, describe the effect of using the word.

fd	bk
rt	lt
to	cs *or* cg
repeat	

b. You've also defined some new terms in this language, including PikeToNeedle, Spine, VarSpine, 8point, and Star. Explain these words in the same way you explained the primitive terms.

The procedure for Shape1 is written below.

```
to Shape1 :Num
 repeat :Num [varSpine 25  rt 360/:Num]
end
```

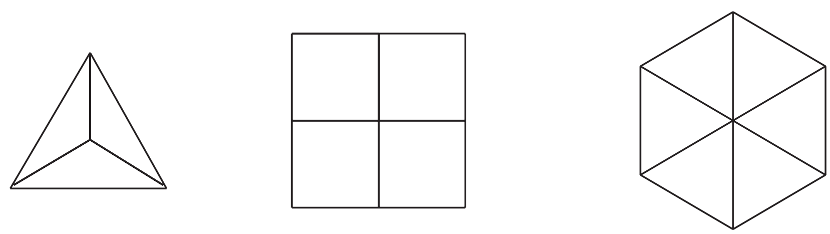
14. Act like a computer turtle. Draw the picture that the command

```
shape1 4 rt 45 shape1 4
```

would produce.

TAKE IT FURTHER. . . . .

15. Use your Star and Shape procedures to make these figures with Logo. Explain what you did.



People often find the hexagon easier than the pentagon: you may want to try it first. It may also help to plan your approach on paper, and then translate your paper-and-pencil method into a computer algorithm.

16. Use your Star and Shape procedures to make *nested* hexagons and pentagons, as in the pictures below. The completed picture should look something like a spiderweb. The smallest of each shape should exactly fit spines of length 10; the next larger should fit spines of length 20, and so on. In each larger shape, increase the spine length by 10 units.

17. Create a new procedure, `Shape3`, that works just the same way as `Shape2` from Problem 5, except that the drawing it produces has a fixed perimeter of 100. The one input, number of sides, must therefore be used to compute both the sidelength and the turn angle. Use `Shape3` to draw a sequence of shapes with increasing numbers of sides. Describe your observations.

```
to Shape3 :Num
 repeat ....
end
```

18. Create `Shape4`, identical to `Shape3` except that the total turning is doubled or tripled (that is, the `360` is replaced by `2 * 360` or `3 * 360`). Again, experiment and describe your observations.

19. **Extended Project** Find a rule for predicting, for any input, the shape that will be drawn by a procedure like `Shape4`, in which the total turning is some (integer) multiple of 360.

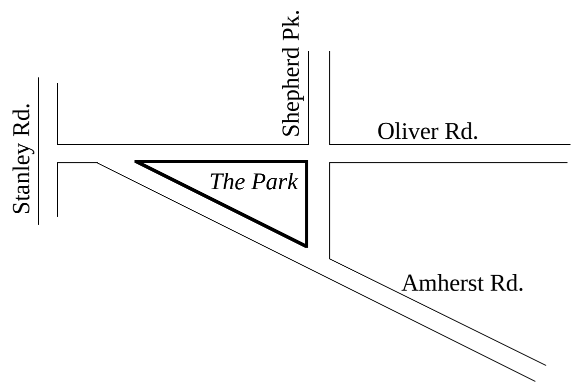
20. **Write and Reflect** Explain how you solved Problem 16 for hexagons. Did you use the same method for pentagons? If not, explain why not, and what you *did* do to solve the problem for pentagons.

Investigation 1.12

Algorithmic Thinking: Irregular Figures

The stars and polygons you've just been drawing are all "regular." That is, their sides and angles are all the same. But life is not always that simple. In this investigation, you will develop a strategy for drawing shapes that are not regular.

1. Sometimes, streets meet in ways that create triangular blocks that cities often use for open space. The instructions below start to draw this triangular park. Your job is to figure out how to complete it.




```
cs fd 100 lt 90 fd 200
```

2. You'll need a fairly large turn to head back toward the starting corner of the block. More than 90°? How big? Experiment until you can complete the procedure below, with the turtle facing north again as it did at the start.

```
to TriangleBlock
 fd 100
 lt 90
 fd 200
 lt ...
 fd ...
 lt ...
end
```

3. What is the total amount of turning (the sum of the three left turns you used) in this block?

Think about the total turning in a star.

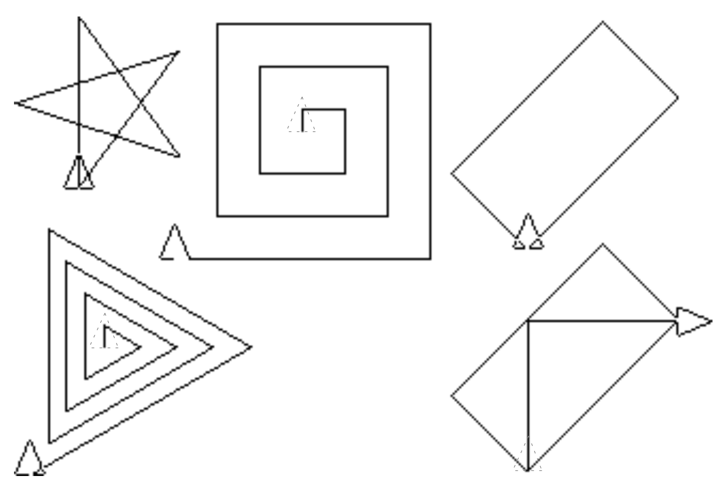
4. Your answer to Problem 3 just required adding three numbers. Explain why your numbers *should* have added up to that sum. That is, why would any *other* sum tell you that one (or more) of your turns was not quite correct?

5. The three angles you turned were exterior angles. Figure out each of the triangle's interior angles.

Turn Command	Exterior Angle	Interior Angle
lt 90	90	90
lt ...		
lt ...		

6. Write a reasonable explanation of why the sum of the interior angles of a triangle must be 180°.

Here are some pictures to draw with the turtle. All the pictures show where the turtle ended. Three pictures show a "ghost" where the turtle started; in the other two, the turtle started and ended in the same place.

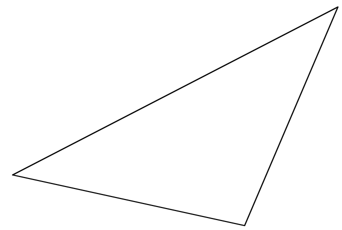


7. Figure out how to make each picture. Clean up your method (getting rid of the errors you correct, the trial steps that helped you figure it out, and any other unnecessary commands), and turn your list of commands into a procedure with a meaningful name.

Investigation 1.13

Constructing from Features: Moving Pictures

Here lies a triangle.

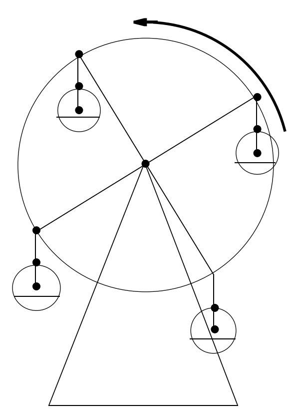


To find out what's true about *this particular triangle* is easy enough. You just study it, measure it, think about it, and list what you find.

But to find out what's true about *triangles* in general is quite another matter. Too many triangles, too little time. One needs another way of investigating.

If only you could stretch and pull and change the diagram of a triangle so that it represented not just one shape but all shapes of triangles. *Then*, studying *that* squirming figure, you might begin to figure out what seems to be true of *all* triangles, regardless of their shape.

In this investigation, you'll learn to use a tool called *geometry software*. This tool is "dynamic" because it lets you animate your moving pictures, bringing your figures to life. With it, you can construct pictures with the *required features* built in, and yet with other features variable in all the ways you need for your experiments. You can build squares that stay square (because you *built in* the features of perpendicular and equal sidelengths), and yet can be rotated or resized at will. You can even animate your pictures: you can invent a carnival ride and set it in motion on your screen!



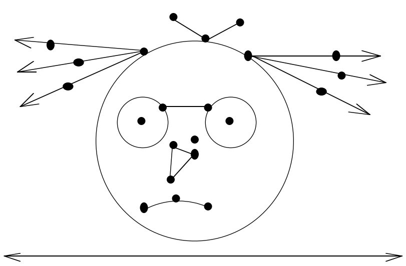
A Ferris wheel

GEOMETRY SOFTWARE'S BASIC TOOLS

Every geometry software program has tools for creating the basic elements of geometric constructions, for labeling these elements, and then for moving them around. Most also have ways of hiding "construction lines," lines that help you *make* the drawing, but that are not really part of the finished drawing.

1. Explore your software until you can do each of the following. Consult your manual or ask questions, if needed.

A little bit of everything:

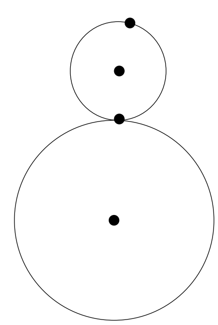


- **a.** Make a triangle out of line segments.
- **b.** Make two circles with a line segment connecting them (place the segment's endpoints *on* the circles).
- **c.** Move a point, segment, or circle in each of these first two drawings.
- **d.** Make a ray.
- **e.** Make a line.
- **f.** Make a point that travels *only* along a segment.

2. a. Using the point tool, place two points on your screen roughly like this:

Then, using *only* the circle tool, complete this picture. Make certain that your picture *never* contains more than four points.

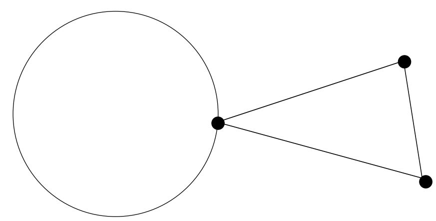
Don't settle for constructing this figure with, say, five points and then erasing one! Find a way to construct it with four points.



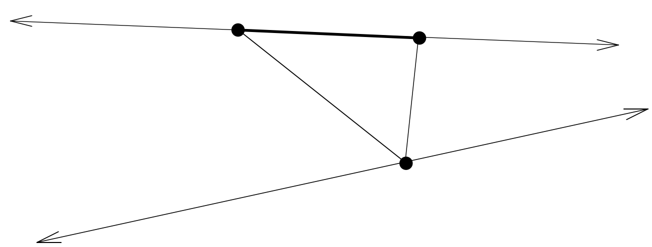
b. Now, move each point around and describe the effect on the drawing. You may want to use a labeled sketch to help identify the points you are describing.

CHECKPOINT

3. Create a triangle that has two vertices that can be moved about freely, and one that can be moved only on a circle.



4. Draw two lines, and construct a triangle that has two vertices that can be moved, but only along one of the lines, and one vertex that lies on the other line.



DRAWINGS *VS.* CONSTRUCTIONS

To make moving pictures that stay the way you want them, you must *build in* the necessary features. Points that are to be *on* a line or circle cannot be created first and then adjusted to look right. They must be *created on* the line or circle. Lines that are required to pass through a point must *use* the point as part of their definition. (The rest of the definition might be a second point or a slope, or perpendicularity to another

line, or . . . .) Squares that are to stay square must have perpendicular sides (and at least one other property) built into them.

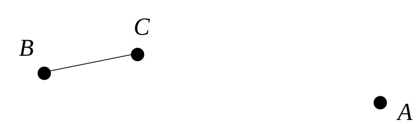
As you construct a figure, you can tell the software to build in the properties that you want. Then, if you drag around one part of your figure, all the other parts will adjust accordingly. Parallel lines will remain parallel, midpoints will remain midpoints, and so on. When a dynamic picture is adjusted to *look* right but does not have its essential properties built in, we say it is “drawn” instead of “constructed.”

CONSTRUCTING A WINDMILL

Step 1: Get a New sketch page on your computer.

Labels may not appear automatically in your software, and if you create things in a different order or make extra points along the way, your labels may not be the same as the examples shown here. Many software tools allow you to *change* the labels any way you like.

Step 2: Make a point A, and a segment $\overline{BC}$.



Your screen should now show four objects: three points A, B, and C, and a segment $\overline{BC}$.

Step 3: Use the point A and the segment $\overline{BC}$ to make these three constructions:

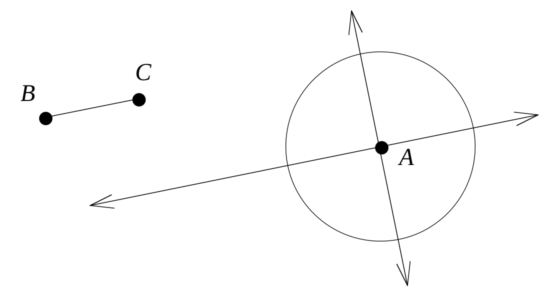
Does a line perpendicular to $\overline{BC}$ have to touch $\overline{BC}$?

1. Use the appropriate tool to *construct* a line, through point A and perpendicular to segment $\overline{BC}$, that will remain perpendicular to $\overline{BC}$, no matter how points A, B, or C are moved.

What’s the difference between “a circle with radius BC” and “a circle whose radius is $\overline{BC}$”?

2. *Construct* a line, through point A and parallel to segment $\overline{BC}$.
3. *Construct* a circle that has point A as its center, and has its radius defined by segment $\overline{BC}$. Stretching or shrinking $\overline{BC}$ should cause the size of the circle to

stretch or shrink to keep its radius equal to *BC*. Again you will need a special tool for this construction.



All three constructions

Step 4: Keep this sketch! You will need it again soon.

5. Use the selection tool to drag *A*, *B*, and *C* around. Describe what happens.

6. No geometry software allows you to construct a perpendicular to a line (or a segment) without identifying both a line *and* a point. Why is that a sensible restriction?

Hiding Construction Lines

These hand-drawing activities were in Investigation 1.4.

Sometimes you need to use a line or circle to build a figure, but you don't want to see these "construction lines" in the finished product. When you used construction lines in making hand drawings, you later erased them. *With computer sketching, you must not erase construction lines*, but you can *hide* them. Find out how to hide parts of a figure with your software.

Improving the Windmill

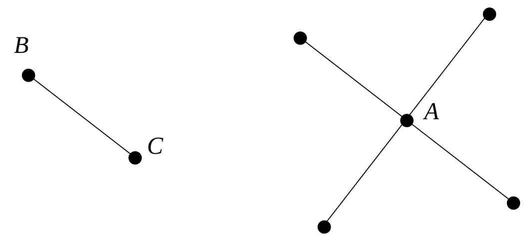
Using the sketch that you created above:

Step 1: Put points where the circle intersects the two lines.

Step 2: Construct segments from the center *A* of the circle to each of the intersection points.

Step 3: Hide (*don't delete!*) the circle and lines. Don't hide the segments.

Crank the windmill!



Steps 1–3 completed

7. **a.** Describe what happens to the segments when you rotate $\overline{BC}$.

b. Describe what happens to the segments when you stretch or shrink $\overline{BC}$.

8. **Beautiful and Surprising** Most geometry software provides a way to Trace the position of objects as they move. Turn the Trace feature on for the four points at the end of your windmill. Then move point *B* about, and watch what happens. Describe the effect.

9. Use the geometry software to draw two intersecting segments. Adjust them until they appear to be perpendicular to each other and of roughly equal length. Move around one of the endpoints. How does the figure behave, compared to the construction you made above?

DRAWING UNMESSUPABLE FIGURES

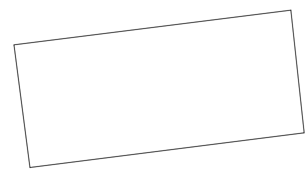
A way to think about the difference between a construction and a drawing is to ask what properties are guaranteed to remain when points or other parts are moved. A square that you *draw* (not construct) is guaranteed to remain a quadrilateral. It may happen, at the moment, to look like a square, but it can be changed from a square by dragging a vertex or side. Its squareness is not guaranteed.

You may want to hide (*not delete!*) construction lines that you need along the way. Why not delete? To get two segments perpendicular, for example, you might create a *Perpendicular Line* and then lay a segment on it. Once the segment is properly placed, you can *hide* the line, and the segment will still be tied to it: as the line moves, the segment will move, too. But if you *delete* the line, who knows what the segment will do?

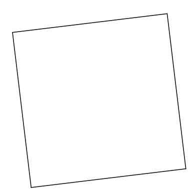
In each of the puzzles below, your mission is to construct pictures that are guaranteed UnMessUpable. Nobody should be able to mess your picture up by dragging a point or segment around. If any part is moved, the other parts must adjust so that the figure will still be "what the customer requested." Work to create the thing that is specified, not just something that looks like it. Customers grumble when their squares turn oblong.

What properties must you build in to guarantee that your figure remains a rectangle?

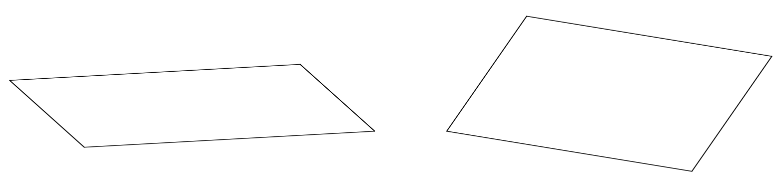
10. Construct a guaranteed rectangle. The customer wants to be able to change its height, width, or orientation when a point or side is dragged, but *it must remain rectangular*.



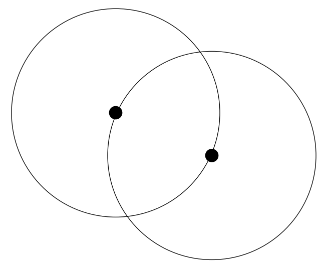
11. Make a square that stays square.



12. Make a figure that can be changed into all sorts of parallelograms, but *only* parallelograms.

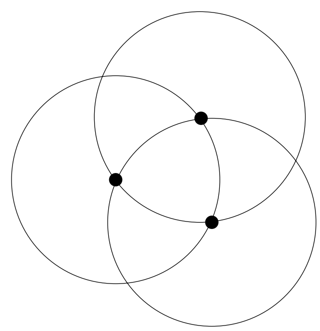


13. Draw two circles through each other's centers. As you change the size of one, the other should change with it.

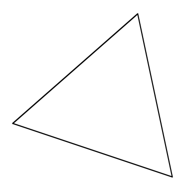


Can you explain why the three points shown in this figure are the same distance from each other?

14. Draw three circles, each through the center of the other two. Changing the size of one should change all three.

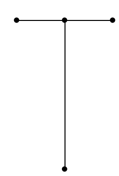


15. Build an equilateral triangle that can be adjusted in size and orientation, but that stays equilateral.

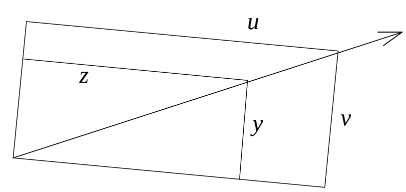


Can you make your T so it always stays upright?

16. Draw the letter "**T**". As a person drags points, the top must remain perpendicular to the stem and centered on it.



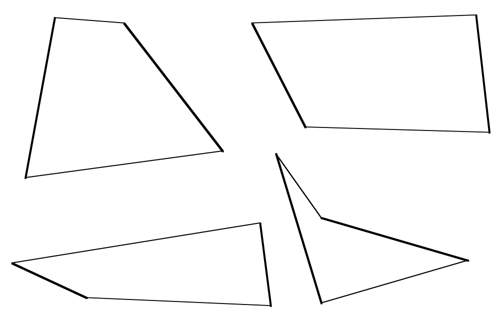
17. a. Make two rectangles with their sides lined-up and their diagonals along the same ray. Make sure the rectangle vertices are built to be on the ray, no matter what you drag around.



b. Your software provides a way to compute the ratio of the lengths of two segments. Use that feature to compare the two ratios $\frac{z}{y}$ and $\frac{u}{v}$.

18. Challenge Invent a kind of quadrilateral that can be deformed in all sorts of ways, but *always* has one pair of opposite sides equal in length but not necessarily parallel.

The heavy lines are the opposite sides that are equal in length.



Can you set up *both* pairs of opposite sides to be equal in length but not parallel?

19. Write and Reflect Select one of the UnMessUpable figures you constructed, and write a detailed description of the construction process. To test your directions, swap them with a partner. Each of you should be able to follow your partner's directions, step-by-step, and create precisely the desired figure (no construction lines, etc.) on the screen. Did you both get the expected results?

A SCAVENGER HUNT

Problems 10 through 18 challenged you to construct figures that maintain certain specified geometric relationships. All of them could be solved with parallels, perpen-

diculars, and circles, but many other wonderful things lurk among your software's tools.

You'll have to poke around a bit to find all the items here. Look at all the tools and all the different options for each tool, and look under all the different menus.

20. See how many of these you can find and use. Describe what tool to use and what objects you must select to build each item. Describe a situation in which you might use each one.

a. Construct the midpoint of a segment.

b. Construct the perpendicular bisector of a segment.

c. Construct an angle bisector.

d. Create a polygon or polygon interior.

e. Measure the distance between two points.

f. Measure the area of a polygon. (With most software, you must first designate the region as a polygon or construct the polygon's interior.)

g. Measure the slope of a line, ray, or segment.

h. Calculate the sum of two measures.

i. Construct a copy of an object, rotated by a fixed angle of 45° around a given point.

j. Construct a copy of an object, rotated by a variable angle around a point.

k. Calculate the ratio of two lengths.

l. Construct a copy of an object, reflected through a given line.

21. Find one other tool and investigate it. Describe how to use it and what it does.

CHECKPOINT

22. While you were learning to use the software, you also did some geometric thinking. List some geometric ideas, terminology, or techniques that you learned, relearned, polished up, or invented.

What is the definition of a rhombus? Is there more than one way to define it?

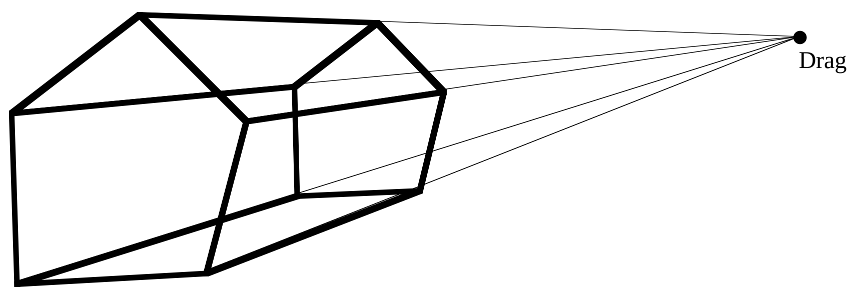
23. Find two different ways to construct a guaranteed unmessupable rhombus with geometry software. Write clear directions for each construction. If either construction is specialized so that it makes only *certain* rhombic shapes but not others, explain how you did that.

24. Look back at Problem 15. How did you make sure that the triangle you built was equilateral? Describe what features of the construction or the resulting figure guarantee that the triangle has equal side lengths.

TAKE IT FURTHER

Choose one or more of these projects.

25. Construct a a picture of a house so that it looks as if it is in perspective. You can build it so that as the points on the "near end" of the house are adjusted to make it look less crooked, the corresponding points on the far end adjust automatically. The point marked *Drag* should be able to move around, changing the house to look as if it is being seen from different perspectives.



WAYS TO THINK ABOUT IT

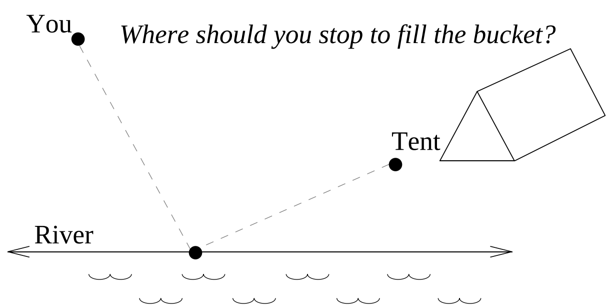
When you drew a house in Investigation 1.2, the two "bases" were identical, their corresponding edges were parallel, and the segments that connected corresponding vertices were all parallel to each other. Here, the connecting segments are not parallel, but instead converge to a point. The bases are no longer identical (what mathematics calls "congruent"), but they are the *same shape* (what mathematics calls "similar"). Their corresponding edges are still parallel.

These facts suggest one of many strategies for constructing the picture. Sketch the near end of the house; create the "drag" point (what both mathematicians and artists call a "vanishing point"); and connect the front vertices to the vanishing point. Then, construct the far end of the

house making sure that each segment of it is parallel to the corresponding segment of the near end.

The Connected Geometry module *Optimization* is all about geometric ways of thinking about problems like this.

26. A good bit of mathematics is devoted to "optimization" problems: finding the shortest path, the lowest cost, the greatest benefit, and so on. Here is an optimization problem that you can investigate with geometry software.



You're on a camping trip. While walking back from a hike, you see that your tent is on fire. Luckily you're carrying a bucket and you're near a river. Where along the river should you get the water to minimize the *travel* it takes to get back to your tent? Justify your answer.

WAYS TO THINK ABOUT IT

By drawing the situation, measuring and adding together the segments of your walk, and experimenting, you can find where the best spot is, but what could you possibly *say* about it? Is there a pattern that allows you to predict where the best spot will be? Is there any fact that is invariably true? Or does each new placement of you, the tent, and the river give a new solution with no relation to the previous one?

Here are three approaches that good thinkers use when they are trying to solve difficult problems.

- **Think about "ballpark" solutions rather than worrying about precision.** For a given placement of the tent and you, is the best spot to get water closer to the tent or closer to you?

- **Think about how the solution depends on one feature of the problem.** In this problem, for example, you might think about how the solution changes as, say, the tent's distance from the river changes. If the tent were very close to the river, how would that affect the best place to run? If the tent were very far from the river, how would *that* affect the best place to run?
- **Try to make an analogy with a related situation that you understand well.** In this case, that might mean thinking about what you know about "shortest distances." For example, one thing that we all assume is that the shortest distance between two points is along a straight line. You might try to find some way of looking at this problem that makes use of this idea.

27. Draw a picture, for example, a person, animal, carnival ride, or machine, using geometry software. Find a way to get the tail to wag, or the eyes to roll, or the ride to move as another part of the picture is moved or animated.

28. Use geometry software to construct a triangle that has sides that are adjustable, but that have unvarying proportions: one side is 2 units long, the other two sides are 3 and 4 units long.

Investigation 1.14

WARM-UPS

You already know a great deal about invariants. The word has already been used about a dozen times in this module. You may have even used it yourself.

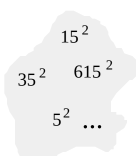
"What kinds of invariants should I be looking for in geometry? What strategies will help me find them?" This section of the module begins to answer these questions, and gives you some practice.

Geometry software can often be a big help. By stretching and shrinking various parts of a figure, you can get a feel for how the parts "work" together, what patterns exist, and what stays the same as the details change.

The invariants you find by experimentation in this investigation might become useful theorems.

Visualize; draw pictures; make calculations; do whatever helps you make an educated guess. For now, don't worry about proving that your guess is true.

Situation 1: Collections of Numbers Something that's true of each thing (or each pair of things, or triple, . . .) in a collection is an *invariant* for the collection.

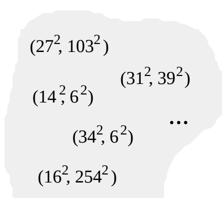


1. Here are three different collections to study.

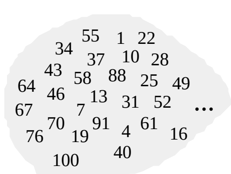
a. One set contains squares of numbers that end in 5. Perform the calculations to see what these numbers "look like." What invariants do you find?

Invariants over a set are things that are the same about each member of the set.

b. Another set contains *pairs* of square numbers. For each pair, the numbers before squaring have final digits whose sum is 10. What invariants do you find in the set of pairs of squares?

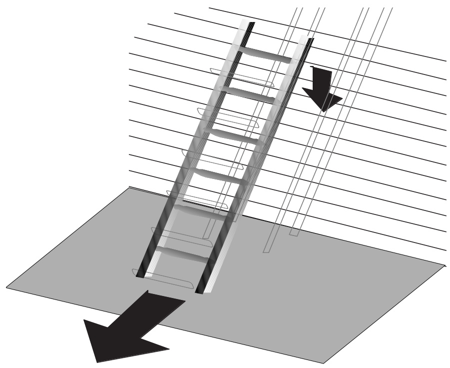


c. Decide whether or not 301 is in the set {1, 4, 7, 10, 13, 16, 19, 22, . . . } and give a reason for your decision. **Extra challenge:** What seems to be true of the product of any two numbers chosen from the set? What seems to be true of the sum of any four numbers from the set? What about the difference of any two numbers?



In this problem, your *reason* should contain a statement of an invariant!

Situation 2: A Ladder Leaning Against a House Something that stays the same while things around it change is an invariant for that situation.



If a ladder is set on mud, gravel, snow, or ice, its base can slide away from the house. As that happens, the top of the ladder slides down the side of the house. This situation involves just a few objects, but many things change as the ladder slides.

A "model" is *very* helpful in studying this problem. A geometry software model is especially convenient.

Approach these problems as experiments.

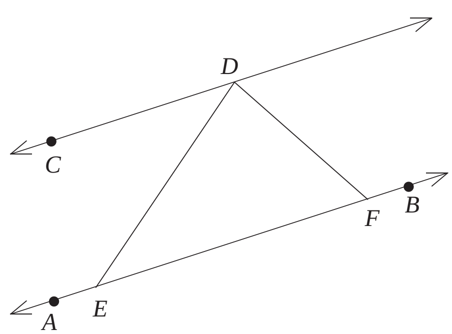
- **Draw and measure the objects with geometry software.**
- **Drag parts of the figure; watch what changes and what remains the same.**
- **Make conjectures that seem likely; find a way to test them.**
- **Organize and record your results.**

2. List everything that you can think of that *changes* in this situation, and everything that seems to be *invariant*. Discuss the lists with your classmates.

Situation 3: Geometric Objects Positions of points, intersections of lines, lengths of segments, angle measures, or even sums or ratios of these measurements may be invariant.

3. With geometry software, create a circle and its diameter. Measure the circumference, the diameter, and the area. Also calculate ratios of each pair of measurements. What seems invariant as you change the size of the circle?

4. Create two parallel lines a fixed distance apart. Create $\triangle DEF$ with points D, E, and F on the lines as shown.



"$m\angle D$" is read "the measure of angle D."

What seems invariant in $\triangle DEF$ as D moves along its line?

a. the measure of $\angle EDF$

b. $DE + DF$

c. the perimeter of $\triangle DEF$

d. the area of $\triangle DEF$

e. the sum of the measures: $m\angle EDF + m\angle DEF + m\angle DFE$

Can you find any invariants that aren't listed above?

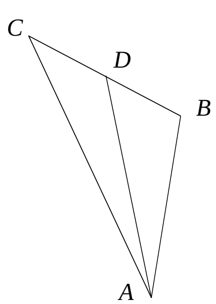
5. Continue your study of the figure from Problem 4. As D moves:

a. Find a pair of angles that have equal measures no matter where D is placed on the line.

b. Find pairs or groups of angles that have an invariant sum of 180°.

c. Is there a pair of angles where the measure of one is always greater than the measure of the other?

What is a *median*?



6. Draw $\triangle ABC$. Construct the (guaranteed!) midpoint D of side $\overline{BC}$. Draw the median $\overline{AD}$. As you stretch and distort $\triangle ABC$, what remains invariant? (Be sure that point D remains a midpoint!)

a. Find segments whose lengths are in constant ratio.

b. Are there any invariant areas? Ratios of areas?

c. Find at least one other invariant. Provide a chart or table of measurements and some sketches to demonstrate the measures or ratios that have not changed.

FOR DISCUSSION

The word *median* is also used in statistics. In what ways are the meanings alike?

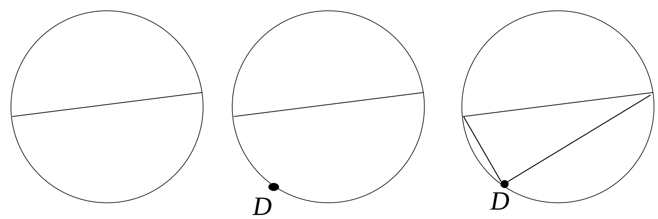
Even *outside* of mathematics, it is important to look not only for what changes, but also for what doesn't change. Finding a nontrivial invariant—something that otherwise different cases have in common—is often the key to a deeper understanding of a situation or phenomenon. Try to find examples from history, psychology, literature, or music.

Investigation 1.15

Numerical Invariants

Constant Measure

Using geometry software, construct a circle and one of its diameters. Place a new point on the circle, and complete the triangle as shown below.



1. Try moving D around the circle. What measures or relationships are invariant as D moves? Look at angles, lengths, sums, and ratios.

2. Now try leaving D in one place and stretching the circle. What measures or relationships remain invariant as the size of the circle changes?

Searching for Patterns

Each table below and on the next page contains four pairs of values.

c	d
2	8
4	10
5	11
7	13

e	f
2	8
3	7
5	5
6	4

x	y
2	$2\frac{1}{2}$
4	$1\frac{1}{4}$
5	1
10	$\frac{1}{2}$

a	b
45	135
72	108
94	86
144	36

w	*z*
2	8
3	$5\frac{1}{3}$
4	4
5	$3\frac{1}{5}$

g	*h*
2	8
4	16
6	24
9	36

m	*n*
10	25
36	90
40	100
50	125

p	*q*
15	105
30	120
45	135
72	162

3. In each table, there is an invariant relationship between the numbers in the pairs. That is, for each pair of numbers, you can compute a third number that's the same for all four pairs in the table. In *these* tables, the simplest invariant relationship will be a constant sum, difference, product, or ratio. Find an invariant relationship for each table.

4. In Problem 3, you found very specific invariants. More generally, there are only two patterns for the numbers in the pairs:

- The values change in the *same direction*: both increase or both decrease.
- Values change in *opposite directions*: one increases as the other decreases.

a. For each table, find which pattern fits.

b. For each pattern, which of the operations, $+$, $-$, $\times$, or $\div$, helped you compute an invariant? Explain why one should *expect* those particular operations to fit with those particular patterns.

5. In this table, too, there's an invariant relationship between the numbers in the pairs. Find a computation that shows the invariant relationship. Do the numbers in the table change in the same direction or in opposite directions? Change the statement about "only two patterns" from Problem 4 to account for this situation.

a	*b*
1	-2
2	-4
5	-10
12	-24

6. Choose one type of numerical invariant (constant sum, difference, product, or ratio) and create a table that follows the *same direction* pattern.

7. Create an *opposite direction* table that does not seem to have a numerical invariant.

8. Here's a table that isn't a simple $+$, $-$, $\times$, or $\div$ relationship. The numbers are deliberately messy, too. It will take some trying, but you can find an invariant.

r	*s*
45	114
60	151.5
72	181.5
108	271.5

CONSTANT SUM AND DIFFERENCE IN GEOMETRY

On the computer, draw a line containing points A and B. Place C on the line, between A and B. Measure segments $\overline{CA}$ and $\overline{CB}$, and keep track of their lengths in a table, like this:

CA	*CB*

9. Keeping points A and B fixed, move point C back and forth between A and B. Do the measured lengths change in the same or opposite directions? Compute a numerical invariant. To what does this number correspond?

10. Now experiment with locations for C on the line that do not lie between A and B. Do the measured lengths change in the same or opposite directions? Make a new table and inspect your data. What is the invariant now?

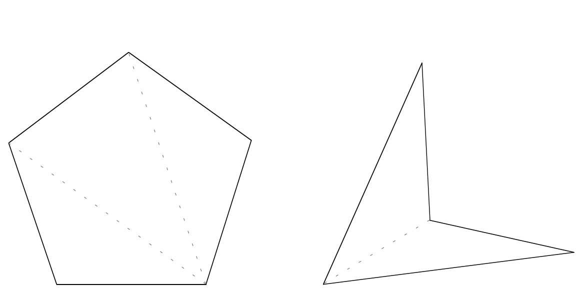
You may already be convinced that the sum of the angle measures in a triangle is invariably 180°. What about the sums of angle measures for other polygons? Will they *all* be 180°, or might each type—trapezoids, parallelograms, rectangles, pentagons,

hexagons—have its own special fixed number? If so, is there some way to predict, given a type of polygon, what the sum of the angle measures will be?

11. Experiment on quadrilaterals, pentagons, hexagons, etc. Be sure to check both regular and irregular shapes. Which groups of polygons have constant sums of angle measures?

How can one possibly show that this is really true for *every* polygon? Can you find a convincing argument?

Reasoning About the Results Every polygon with n sides can be cut up into $n - 2$ triangles.



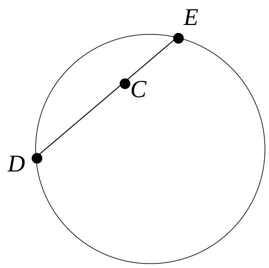
3 triangles from a pentagon *2 triangles from a quadrilateral* *6 triangles from an octagon*

12. Write and Reflect Assume that the angle sum in a triangle is invariant. Use that fact to write an argument that for n-sided polygons, the angle sum is also invariant. Find a rule that will tell you the angle sum if you know the number of sides.

CONSTANT PRODUCT IN GEOMETRY

With geometry software, make a circle and a point C anywhere inside it. Place D on the circle. Construct a line through D and C, to create a new intersection E. Mark that intersection, hide the line, and construct segments $\overline{DC}$, $\overline{EC}$, and $\overline{DE}$.

A segment whose endpoints are on a circle is called a *chord*.



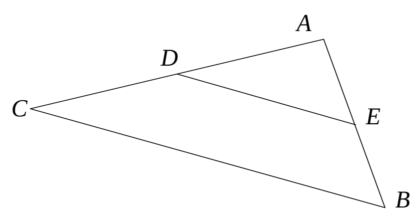
As you move D along the circle, $\overline{DE}$ will pivot about C.

13. Measure $\overline{CE}$ and $\overline{CD}$. As the chord pivots about C, note how CE and CD vary. Do these measurements change in the same or opposite directions? Use that information to guide you in computing a numerical invariant.

14. The number you found did not depend on the location of D—you could move D, and this number remained fixed. But the number does *not* remain fixed as C is moved. For what location of C inside the circle is this number largest? Why?

Constant Ratio in Geometry

With geometry software, draw a triangle, and then construct and connect the midpoints of two sides. Your construction will look something like the one pictured here, where D is the midpoint of $\overline{AC}$ and E is the midpoint of $\overline{AB}$.

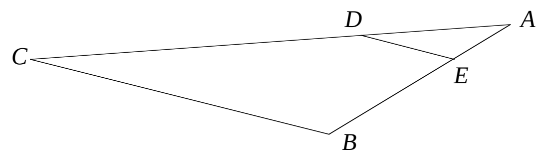


15. Change the triangle's shape by moving one of its vertices. As you change the triangle, what stays the same? List at least three invariants.

16. Measure the lengths of $\overline{DE}$ and $\overline{BC}$. Compare these lengths as you change the shape of the triangle. Are they "same" or "opposite" changers? Use calculations to conjecture the exact relationship between these two segments.

Draw a new $\triangle ABC$, and place a point D arbitrarily on $\overline{AC}$. Through D, construct a parallel to side $\overline{CB}$. Use that line to construct $\overline{DE}$, and then hide the line.

Your construction resembles the earlier one, but this time D and E are movable points rather than midpoints.



17. As D moves along $\overline{AC}$, $\overline{DE}$ moves with it. Look at lengths and areas, and try to find some invariant relationships. Record your conjectures and appropriate supporting evidence.

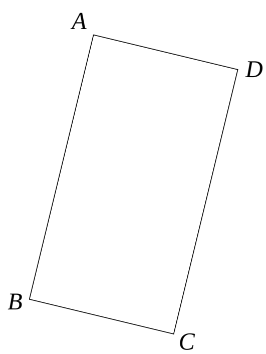
WAYS TO THINK ABOUT IT

Use the familiar strategy of looking for "same-direction changers" and seeing if their relationship is constant. There might also be interesting "opposite-direction changers."

This is a good example of slightly changing a problem. From Problem 15, you already have some ideas about a special case—when D is located at the midpoint. Some ratios were invariant in that case. If D is in a different *fixed* location and the vertices A, B, or C are moved, are those ratios still invariant, or did their invariance depend on D being precisely at the halfway mark?

As D moves, do the ratios remain constant? If so, you have an invariant again. If not, perhaps there is a relationship between two or more of the ratios.

CHECKPOINT



18. Construct a (guaranteed!) rectangle *ABCD* that can be stretched to any length or width. Which of the following are invariants?

a. The length-to-width ratio: $\frac{AB}{BC}$

b. The ratio of the lengths of the opposite sides: $\frac{AB}{CD}$

c. The perimeter of rectangle *ABCD*

d. The ratio of the lengths of the diagonals: $\frac{AC}{BD}$

What other numerical invariants can you find here?

19. Construct a triangle that has one right angle, even if you alter other parts of it. Construct the midpoint of the hypotenuse (the side opposite the right angle). Measure the distances from the midpoint of the hypotenuse to each vertex. Look for an invariant and describe it.

TAKE IT FURTHER

20. On paper (or on the computer), draw two points, A and B, which never move. Make the distance between them no longer than 6 units (your choice of units). Imagine a point P that can move along various paths. Each problem below describes one of these paths in terms of P's relationship to A and B. Your job is to describe what P's path *looks* like in each case. Draw a picture or describe it in words.

a. As P moves along this path, PA always equals PB. What is the shape of the path?

b. This time P's path keeps $PA = 5$.

c. As P moves along this path, $PA + PB = 6$.

d. $m\angle APB = 90^\circ$, no matter where P is located along this path.

e. $m\angle APB = 30^\circ$, no matter where P is located along this path.

f. $m\angle ABP = 30^\circ$, no matter where P is located along this path.

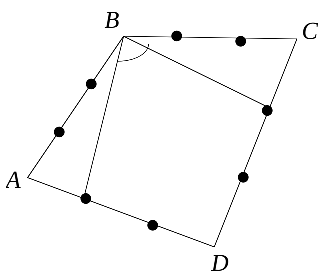
Investigation 1.16

Spatial Invariants

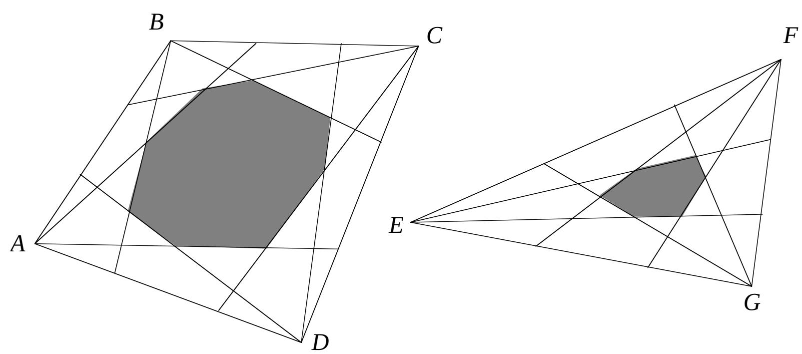
Shape: A Geometric Invariant

Number of Sides

To connect using the widest possible angle, use the nearest third-points not on the same segments as the vertex.



The pictures below show a quadrilateral and a triangle. Each edge of both polygons is divided roughly into thirds, and each vertex is connected to two of these "third-points" at the widest possible angle. The connecting lines surround a region. From the picture, it would appear that when the outside shape has four sides, the inside shape has eight, and when the outside shape has three sides, the inside shape has six.



If you want to be precise, you can learn, or make up a way, to subdivide a segment accurately into thirds. For this construction, however, it is OK to subdivide the segment "by eye" (or by measurement). Using software, your "approximate thirds" will stay as accurate throughout the experiment as they were at the start.

Is this a reliable pattern? That is, does the inside shape *always* have twice the number of sides as the outside shape when vertices are connected to "third-points" in this way?

1. Experiment with constructions that you can stretch. Look at special cases: for example, only triangles or only regular outside polygons. And look at general cases. When (if ever) is there a *shape* invariant?

2. Stating the limits of what *can* happen is often as useful as saying what *must* happen. If the outside polygon has n sides, can the inside polygon ever have *more* than $2n$? If so, what's the largest number it can have? Can it have fewer than $2n$? If so, what's the smallest number? Can the inside polygon ever be *regular*?

Beyond Number

There *was* a shape invariant in the previous experiments, though not as strong a one as the pictures might have suggested. The invariant only had to do with the *number* of sides, and not with the relationship of the sides to one another.

In other words, is there anything besides the number of sides that is invariant here?

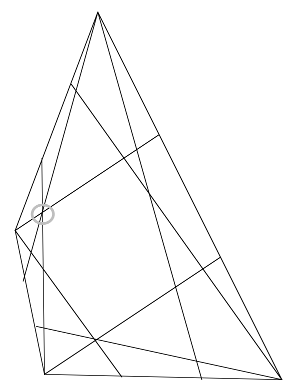
3. Draw a quadrilateral, construct the midpoints of its sides, and connect those midpoints in order.

a. Explain why the inside figure *must* be a quadrilateral.

b. Can the inside figure be *any* kind of quadrilateral, or are certain kinds not possible?

Concurrence: A Geometric Invariant

Here is a picture that might have come from one of the experiments you performed in Problems 1 and 2.



Three or more lines that meet or intersect at a single point are called *concurrent*. Why not call two lines that meet in a point concurrent?

At each vertex of the outside quadrilateral, four lines "run together," or *concur*. That's no surprise; it was intentional.

But it also happens, in this particular picture, that three *inside* lines run together. In fact, this figure shows two such concurrences. (One is circled. Find the other.) These concurrences were *not* deliberately built in; they *are* something of a surprise.

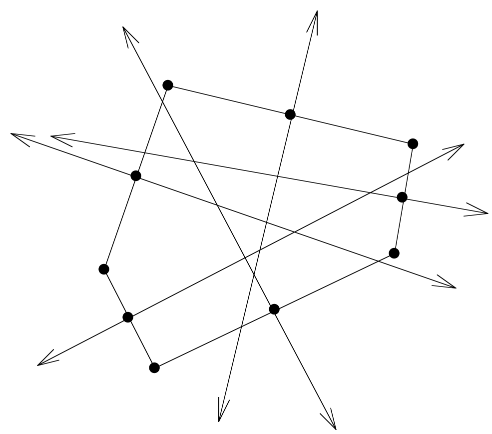
4. Are the concurrences *invariants* for this construction? That is, if the vertices of the quadrilateral are moved, will there *always* be two internal points at which three lines meet?

When concurrence happens *reliably* in a figure—that is, when it is an invariant for that figure—it signals that something special is going on. As you work through the experiments in this section, keep careful track of what you find because you will need

them for the last problem in the section, which asks you to organize your observations into a report.

5. Your teacher will provide a page with pictures of several different regular polygons; draw all the diagonals for each of them. For which polygons are there concurrences among the diagonals? What conjectures can you make for *other* regular polygons?

6. Using geometry software, place five points and connect them with segments to create an arbitrary pentagon. (Adjust the points, if necessary, to make your pentagon convex.) Construct the perpendicular bisector of each side of your pentagon.



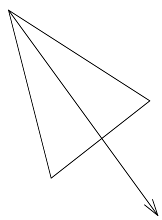
No concurrence

a. Is there a point at which three or more perpendicular bisectors are concurrent?

b. If not, is it possible to adjust the vertices of the pentagon to *make* three bisectors concur at a single point?

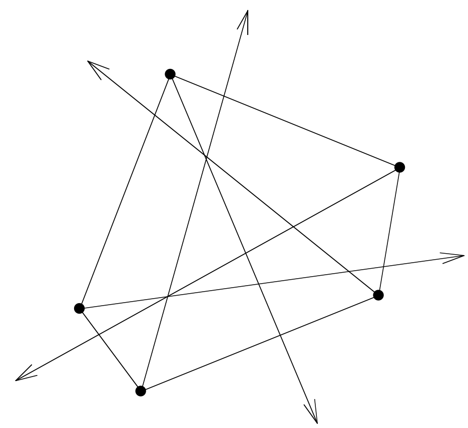
c. Is it possible to adjust the vertices of the pentagon to make *all five* bisectors concur at a single point?

What is an *angle bisector*?



7. Try the same kind of experiment with *angle* bisectors. Start with an arbitrary pentagon, and construct an angle bisector at each vertex.

Two sets of three angle bisectors concur at two separate points, but all five do not concur at a single point.



Is it possible to adjust the vertices of the pentagon to make *all five* angle bisectors concur at a single point?

8. Now try the same kinds of experiments with triangles. Make a triangle and construct the perpendicular bisectors of all three sides. Can you adjust the triangle so that all three perpendicular bisectors are concurrent?

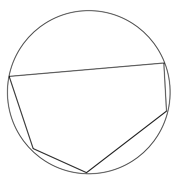
9. Hide the perpendicular bisectors and construct angle bisectors at all three of your triangle's vertices. Can you adjust the triangle so that all three angle bisectors are concurrent?

10. Under what special circumstances (if any) might it be possible for all three perpendicular bisectors and all three angle bisectors to concur at the same point?

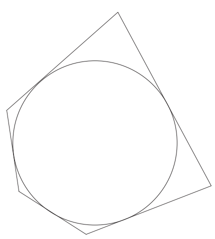
Starting an investigation with special cases is often a good idea. It simplifies what one has to look at, and so it often leads more quickly to fruitful conjectures. Among polygons, the triangle is special—it is the simplest. The following experiments suggest other special cases.

Because a "concurrency" is defined as a point at which *three or more lines meet*, some people find it wrong to claim that perpendicular bisectors of a regular quadrilateral are concurrent. Why? How do you feel about that situation?

This pentagon is said to be *inscribed* in the circle.



This pentagon is *circumscribed* about the circle.



11. Regularity is a very special case. Is "concurrency of perpendicular bisectors" an invariant for *regular* polygons? Experiment. Be sure to try, at least, regular quadrilaterals (squares), regular pentagons, and regular hexagons. What do your experiments show? Explain why that *should* be the result.

12. What about angle bisectors in regular polygons? Again, experiment, describe a conjecture, and explain your result.

13. Using geometry software, construct a circle, place five points on it, and connect the points to form a nonregular pentagon. Check angle bisectors for concurrence. Check perpendicular bisectors for concurrence. Do you observe any invariants?

14. Construct a circle and build an irregular polygon outside of it, carefully adjusting so that all of its sides are tangent to the circle. Perform the two concurrence experiments again. Do you observe any invariants?

15. **Write and Reflect** Review the results of Problems **??** through **??** and write up your observations, conjectures, and reasoning. Try to capture some of the "flavor" of the search—point out surprises, dilemmas, uncertainties, and rejected conjectures—along with the results about which you feel certain.

COLLINEARITY

Three or more points that fall on the same line are called *collinear*.

A *disk* is a circle and its interior.

Just as it is very special for three lines to intersect at the same point, it is noteworthy when three apparently unrelated points lie on the same line.

For the next problem, you will need various sizes of paper disks. To prepare, find something that will help you trace a circle on a sheet of paper. Locate the exact center

Challenge: How can you accurately find the center of a circle if you trace it from a jar or can?

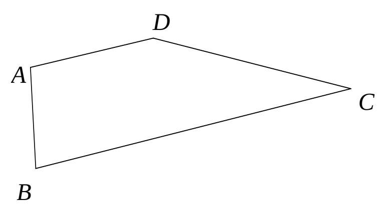
and poke a small hole through the paper right at that point. Carefully cut out your disk. Get together so with four or five of your classmates who have made disks of other sizes, and perform this experiment:

- Draw two points on a large sheet of paper. Place them close enough together so that your smallest disk can touch both.
- Take one of your disks and lay it down so that both points lie exactly on the disk's edge (that is, on the circle itself).
- While your disk is just touching the two points, make a tiny mark through its center onto the paper.
- Remove that disk and do the same thing with the next one.

16. **a.** When you've used all of your group's disks, look at the marks that show where their centers were located. What pattern is there to the arrangement of the centers?

b. Draw two new points. Without using your disks, draw the pattern along which their centers would lie.

17. Construct a (guaranteed!) trapezoid *ABCD* whose vertices and sides can be dragged around.

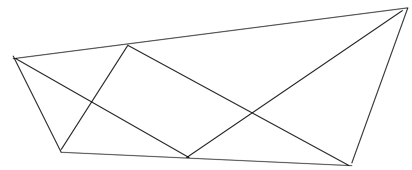


a. Construct the diagonals and their intersection. Also, construct the midpoints of the two parallel sides.

b. Find two intentionally built-in collinearities. Can you find a collinearity that was not intentional? If so, experiment to see if that collinearity is invariant.

18. On paper or computer, build an arbitrary quadrilateral. On one edge place an arbitrary point, and connect it with the two "opposite" vertices. Do the same

with the remaining two vertices and the opposite edge.

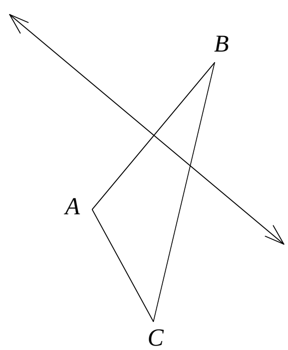


Finally, draw in the diagonals of the quadrilateral. Find two nonsurprising collinearities. Find a surprising one!

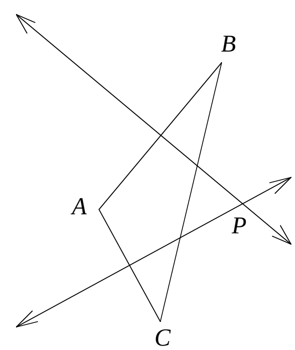
THINKING ABOUT THE RESULTS

When you know how a trick is done, it isn't magic any more!

It may have been surprising that the perpendicular bisectors of the sides of triangles are concurrent. If you analyze the situation, though, it becomes less surprising.



The points on the perpendicular bisector of $\overline{AB}$ are precisely those points that are the same distance from A and B.

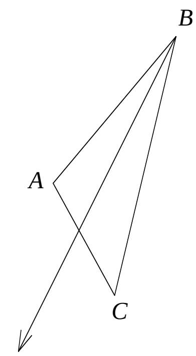


The points on the perpendicular bisector of $\overline{AC}$ are precisely those points that are the same distance from A and C.

19. Write and Reflect Use the drawings and explanations above to describe point P, where the two perpendicular bisectors meet. Why *must* the perpendicular bisector of $\overline{BC}$ also go through point P?

The distance from a point on the angle bisector to $\overline{AB}$ is the length of the perpendicular from that point to $\overline{AB}$.

A similar argument can be made for the angle bisectors.



The points on the angle bisector of $\angle ABC$ are precisely those points that are the same distance from sides $\overline{AB}$ and $\overline{BC}$.

20. Write and Reflect Finish the argument above. Explain why all three angle bisectors in a triangle must meet at the same point.

CHECKPOINT

21. You have run across many special terms. Look through your work and make a list of new terms, with definitions or explanations. If there are words about which you are still uncertain, list them separately.

22. Write what you know about each of these terms. Say what they are as precisely as you can.

diagonal	intersection	inscribed
median	concurrent	circumscribed
collinear	angle bisector	invariant

For now, each of these “rules” will be considered tentative. Many of them will be proved formally later in your study of geometry. Some may turn out to be false.

23. Write and Reflect For each situation you explored in this investigation, choose an invariant that you found, and state it as clearly as you can in the form of a general rule.

Write each rule in a special section of your notebook that contains results that are waiting for further verification.

TAKE IT FURTHER

24. Consider the following statement: "In *any* hexagon with all diagonals drawn in, there can be *at most* one concurrence of three diagonals." Do you think this statement is true or false? Explain your reasoning.

Investigation 1.17

PARALLEL LINES

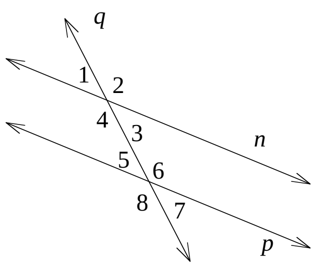
You've seen parallel lines all your life, and they may even have been part of your mathematical studies. By looking for invariants in situations with parallel lines, you will understand more about how they work.

BACKGROUND CHECK

One definition for parallel lines says, "Parallel lines are lines in the same plane that do not intersect." You might also have seen another definition: "Parallel lines are everywhere equidistant."

Another way to say "everywhere equidistant" is "the same distance apart, no matter where you measure."

A line that intersects two or more parallel lines is called a *transversal*. The angles that are formed when transversals intersect parallel lines also have special names.



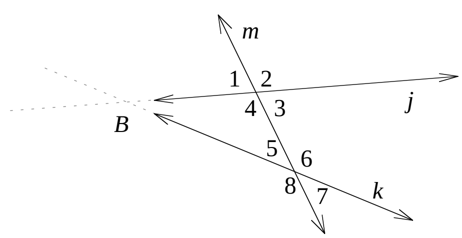
1. Pairs of angles like ∠3 and ∠5 or ∠4 and ∠6 are called *alternate interior angles*.

a. Explain the name. Why "alternate"? Why "interior"?

b. Figure out what "alternate exterior angles" means, and name a pair of alternate exterior angles in the figure.

Pairs of angles like ∠1 and ∠5 or ∠4 and ∠8 are called *corresponding angles*. The angles that are on the same side of the transversal and between the lines (for example, ∠3 and ∠6) are sometimes called same-side interior angles.

The same terms are used when the lines are not parallel. In the figure below, lines *j* and *k* intersect at *B*. Even so, *m* is a transversal, ∠3 and ∠5 are alternate interior angles, and so on.



2. a. Name all the pairs of corresponding angles in this figure.

b. Explain what "corresponds" about corresponding angles.

3. What name could describe an angle pair like ∠2 and ∠7?

EXPLORATIONS

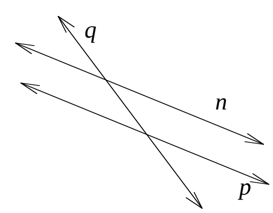
Your construction will look similar to the drawing on the previous page of lines *j*, *k*, and *m*.

4. Use geometry software to construct a pair of intersecting lines along with a transversal. Measure the angles in your figure.

   a. First move the transversal while the other lines remain fixed. What invariants can you find? Look especially for pairs of angles that remain equal in measure and for pairs that have a constant sum.

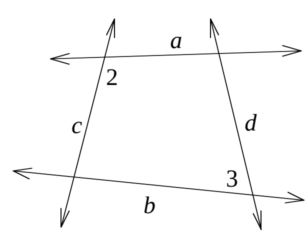
   b. Now try moving one of the intersecting lines while the transversal stays fixed. Record and explain what you find.

5. Construct a pair of parallel lines with a transversal. Which angles stay equal in measure even if the transversal moves or one of the lines moves (while remaining parallel)? What angle sums are invariant? Compare these results to your findings in Problem **??**.



In the illustration here, lines *a* and *b* were drawn carelessly and clearly are *not* parallel.

6. Construct a figure with two parallel lines (like *a* and *b*) and two transversals (like *c* and *d*).



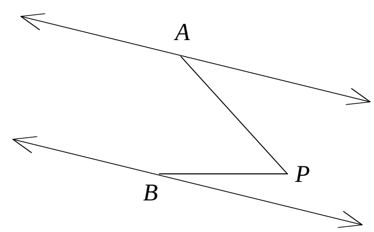
$\angle 2$ and $\angle 3$ are not necessarily equal in measure (in the preceding picture, they are certainly not equal). Move the lines around to make $m\angle 2 = m\angle 3$.

a. When $m\angle 2 = m\angle 3$, do lines c and d have any special relationship? Is that relationship invariant over changes in the (equal) measure of $\angle 2$ and $\angle 3$?

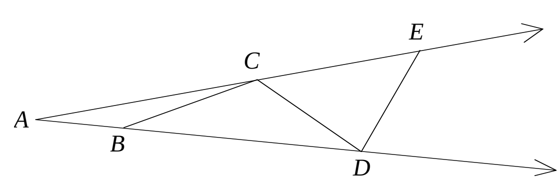
b. If lines c and d are parallel, do the measures of $\angle 2$ and $\angle 3$ have an invariant relationship?

c. How would your answers to the previous two parts of this problem be different if lines a and b were not required to be parallel?

7. Construct a pair of parallel lines with a movable point P between them. Draw $\overline{PA}$ and $\overline{PB}$ to connect the point to the parallel lines. What invariants can you find in this situation? What would you describe as transversals, alternate interior angles, and corresponding angles in this figure?



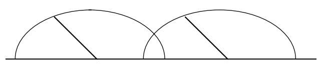
8. Construct a figure with two intersecting lines and points on those lines connected to form three segments. (See the figure below.) Experiment with the figure by moving various parts until you have made $\overline{BC}$, $\overline{CD}$, and $\overline{DE}$ all the same length.



a. If $BC = CD = DE$, what conclusions can you draw about the angles in the figure? Describe any invariant relationships you find.

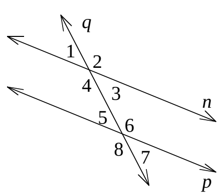
b. If $BC = CD = DE$, is it possible to make $\overline{BC}$ and $\overline{DE}$ parallel? Explain.

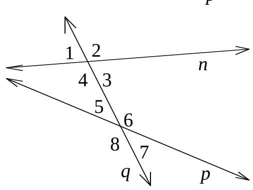
TAKE IT FURTHER...............................



Many mechanical devices use parallel rods to make them work properly. Windshield wipers on some cars and many buses are often made as pairs of parallel rods driving the wiper blades on the windshield of the vehicle. The rods sweep back and forth in a circular motion, but they stay parallel. Trucks and buses sometimes have a wiper system that uses parallel parts in a different way.

9. Study the way parallel parts are used in windshield wipers by looking at several vehicles and then construct physical or computer models of what you saw. What seem to be the advantages of each type? Is it necessary for the wipers to stay parallel? What shapes do the different wipers sweep clean? What factors might make one type of wiper better for a particular vehicle?





10. Use your knowledge of parallel lines to decide which of the following constructions are *impossible*. Explain what makes them impossible.

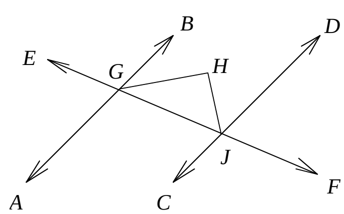
a. n and p are not parallel and $m\angle 3 + m\angle 6 = 180^\circ$.

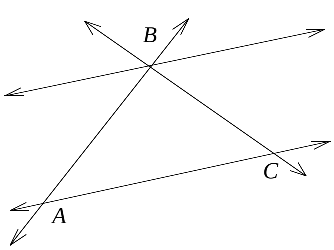
b. $m\angle 4 = m\angle 6$, and n is parallel to p

c. $m\angle 2 = m\angle 5$, and n is parallel to p.

d. $m\angle 4 + m\angle 5$ is greater than $m\angle 2 + m\angle 7$.

e. In the figure below, $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel, $\overline{GH}$ and $\overline{JH}$ are angle bisectors, and $m\angle GHJ < 90^\circ$.





You already have overwhelming evidence that the sum of the angles in a triangle is 180°, you've used it to support other arguments, and you may know a few different ways to argue that it's true. Now you can write a reasoned argument of this fact, using what you know about parallel lines, transversals, and alternate interior angles.

11. Write and Reflect Write an argument that the sum of the angles in *any* triangle is 180°. The picture may give you some ideas.

Investigation 1.18

INVESTIGATIONS OF GEOMETRIC INVARIANTS

Any mathematical investigation is a search for invariants and an attempt to explain them. The following activities suggest five geometric investigations.

Here are some general guidelines for mathematical research.

- **Experiment.** Use hand or computer drawings to help you visualize the situation, explore it, and gather data.
- **Record your experiment.** Describe carefully what you did, and what happened as a result. Explain as well as you can *why* things behaved the way they did. If you have unanswered questions at the end of the investigation, record them, too.
- **Summarize your work.** Write a brief, clear presentation, describing the situation and results, and including whatever drawings you need. List your conjectures, and any important terms, theorems, rules, or ideas that came up in the investigation. Include any questions that require further exploration.

As a reminder, here are some of the invariants you might find.

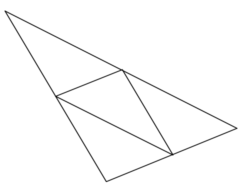
- constant measure
- constant sum
- constant difference
- constant product
- constant ratio
- constant shape
- constant concurrence
- constant collinearity
- parallel lines

MIDLINES AND MARION WALTER'S THEOREM

BACKGROUND CHECK

1. Use geometry software to construct a triangle and all three medians. Change the shape of the triangle by dragging around one vertex or side, and look for invariants as the triangle changes. Consider angles, lengths, and areas, as well as anything else you might find. List your conjectures.

A *midline* (or *midsegment*) connects the midpoints of two sides of a triangle.



This triangle has all of its midlines drawn in.

2. Construct a triangle and all three midlines. Again, consider angles, lengths, and areas. List your observations. Include answers to these questions:

a. From the *length* of one midline, what can you determine about the triangle?

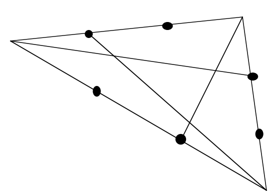
b. If you know the area of the original triangle, what can you say about the areas of other parts of the figure?

THE INVESTIGATION

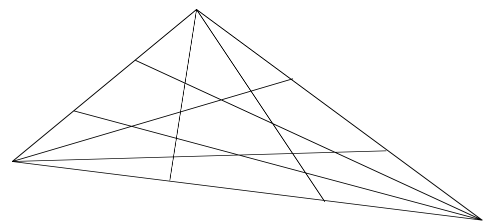
Medians and midlines are results of cutting sides in half. What happens when a triangle's sides are cut into thirds?

Dividing segments is not always easy. If you're not sure how to do it, your teacher should be able to provide some tips.

3. **Situation 1:** Construct a triangle and cut each side into thirds. Connect each vertex to the first trisection point (clockwise) on the opposite side. Change the triangle by dragging vertices and segments. Look for invariants and record your observations.



4. **Situation 2:** Connect the remaining trisection points to the opposite vertices. Change the triangle by dragging vertices and segments. Look for invariants and record your observations.



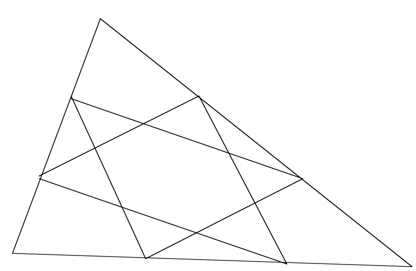
FOLLOW-UP

5. Organize your ideas about cutting-in-half and cutting-in-thirds in a way that allows comparison. For example, all your results about area might be placed together. Look for similarities or patterns—ways to generalize the results.

6. Choose one conjecture from your lists and explain how it changed as the situation changed from cutting-in-half to cutting-in-thirds.

7. Predict how the conjecture would change if you tried subdividing the triangle's sides into fourths, fifths, or some other number of pieces. Test your predictions with an experiment.

8. Use your geometry software to design a triangle with all six trisection points. Connect the first trisection point on one side to the first trisection point on the next side (working clockwise). Then connect the second trisection point on one side to the second trisection point on the next side (still working clockwise).

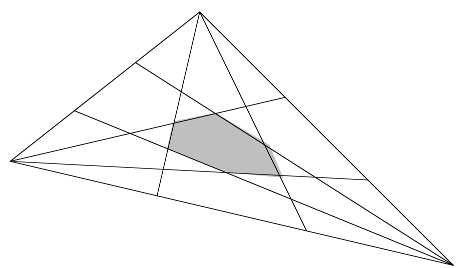


As you change the triangle by moving vertices or sides, what invariants can you find?

PERSPECTIVE: A HIGH SCHOOL STUDENT EXTENDS THE THEOREM

High school students can discover new mathematics. In this essay, you will read about work done by a geometry student that extends Marion Walter's Theorem.

Marion Walter is a professor at the University of Oregon (see Investigation 1.5 for her biography). The theorem that bears her name states a relationship between the areas of the inner hexagon and the outer triangle in the trisection construction you made in Problem 4.



Ryan Morgan was a sophomore at a high school in Maryland when his geometry teacher asked his class to prove Marion Walter's Theorem. Ryan used computer software to help investigate the problem. During his explorations, he made an important discovery that was reported in several newpapers. Ryan gave a presentation about his discovery to a faculty seminar for the Towson University mathematics department. Here is Ryan's description of his experience working on Marion Walter's Theorem.

"One day, my geometry teacher took our class across the hall to our school's computer lab. He wanted us to get familiar with the use of the computers. He did this by having us 'discover' Marion Walter's Theorem on our own. He told us how to draw the triangle, trisect the sides, and draw the hexagon in the middle. It was our job to find something 'neat' about the measurements of the shapes. When I found Marion Walter's Theorem, I got curious, and wanted to see if the same thing worked with any shapes other than triangles.

"This is where all the hard work began. Not having a computer of my own, I was forced to use my school's computers after school. The first thing I did, once I became

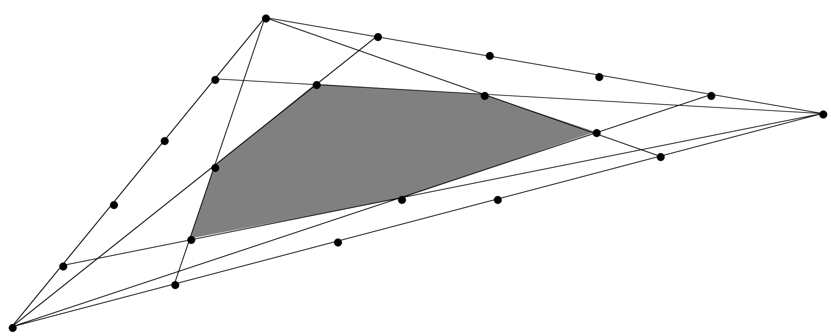
familiar with the software, was to experiment with squares. I attempted to trisect the sides of the square, and see if there was any special relationship there. At first I thought there was. I can't remember exactly what the value was, but it did seem that there was a constant ratio between the square and the octagon that was in the middle of the square. At that point, I really thought I was on to something.

"So, the next thing I did was try the same thing with a pentagon. I trisected the sides, compared areas between the pentagon and the now 10-sided figure inside. . . . There was no constant ratio. I was upset, because at this point I had spent maybe a week or two, every day after school, working on this thing, and now I had hit a dead end. But I didn't give up."

Ryan first tried to extend the theorem by using figures other than triangles, but that didn't produce any invariant. So he returned to triangles and found a different way to extend the theorem.

"This time I concentrated only on triangles, and no other shape. Trisecting it was the whole basis behind Marion Walter's Theorem. . . . [so] I started 5-secting, 7-secting . . . , the triangle."

These new ways of "secting" the triangle, as Ryan called it, also produced hexagons in the center.



"When I compared the area of the triangle and area of the internal hexagon, I noticed a constant ratio between the two. And for every example I tried (odd number 'secting' per side only), I was able to find a ratio that was constant no matter how the triangle was altered in size.

"The next step was to find a relationship between the number of sections per side and the ratio between the areas of the triangle and hexagon. Using the regression

functions on a regular scientific calculator, I was able to do just that, and came up with the formula

$$y = \frac{9}{8}n^2 - \frac{1}{8},$$

where n is the number of sections per side, and y is how many times bigger the area of the triangle is than the area of the hexagon.

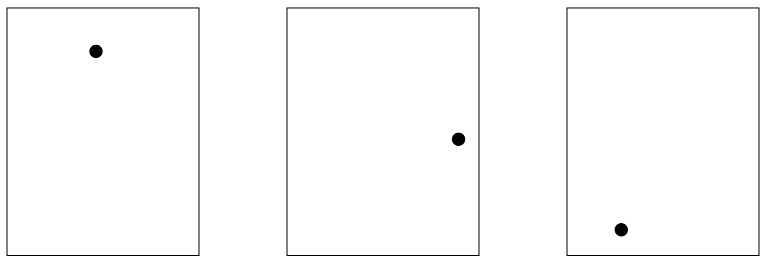
Ryan's conjecture, which he based on his experimental data, has been proved.

"It took a lot of time and effort, but it was worth it. And before I can take credit for my theorem, I must thank Marion for creating her theorem to begin with; without hers, mine never would've come around."

A Folding Investigation

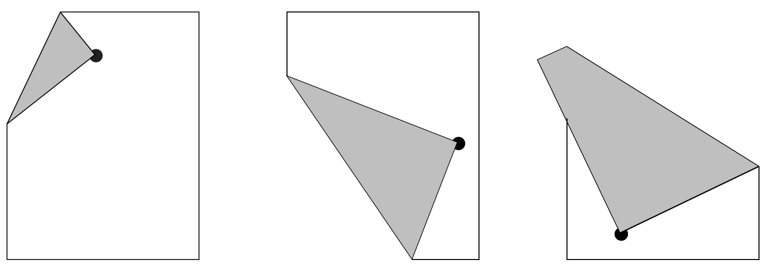
Background Check

9. Mark a point somewhere on a rectangular sheet of paper. For example:

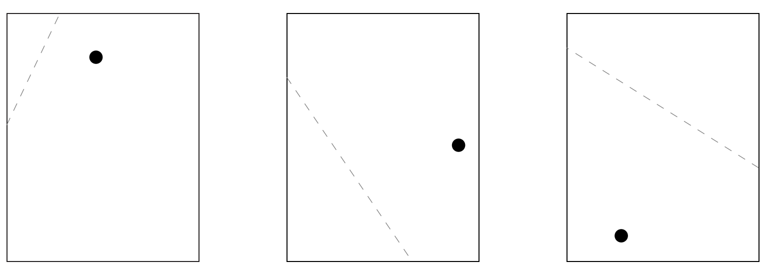


One of these should work . . .

Bring one corner of the paper directly to the point you've marked, and crease carefully. For example:



Now unfold the paper to see the crease you've made. For example:

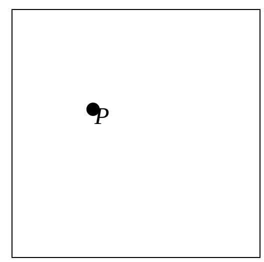


Connecting the point to the corner with a line segment might help you see the relationship.

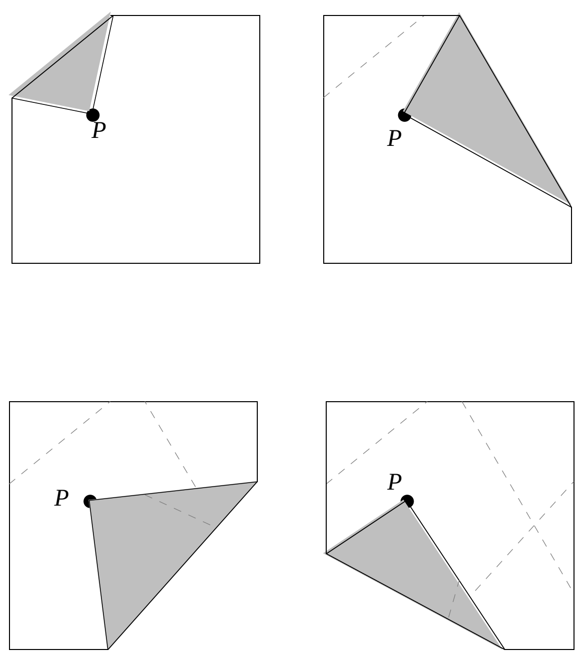
Describe the geometric relationship of the crease line to your choice of point and corner.

THE INVESTIGATION

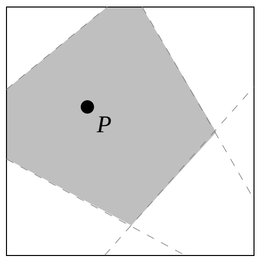
Starting with a *square* piece of paper, mark an interior point *P*.



One at a time (and unfolding in between) fold *each* corner in to touch the point *P*. Crease carefully each time.



When you are done, the creases will divide the square into several polygonal regions. In the example shown here, the region containing the original point is a hexagon.



The sides of the hexagon can be either creases or sides of your paper.

10. Perform the experiment several more times. Does the region containing *P* always have six sides, no matter where you place *P* at the start? If you think it does, give a good reason why that should happen. If you think that there will be some other number of sides, give at least one example to support your claim.

11. Imagine starting with a square, picking some location for *P*, and folding each corner to *P* as described above.

a. Find a good reason why the region that contains *P* must have fewer than nine sides.

b. Find a good reason why the region that contains *P* must have more than three sides.

12. Problem 11 asked you to show that the region that contains *P* cannot have less than four sides, and cannot have more than eight. Perform some experiments that help you narrow the range even more. What are the minimum and maximum number of sides you find?

A good way to answer this question is with a picture—perhaps a map of the square, with one color for all the places *P* can be to produce a hexagon, and a different color for places that produce some other result.

13. How does the placement of *P* determine how many sides the polygon will have? Where can you place *P* to get the minimum and maximum number of sides?

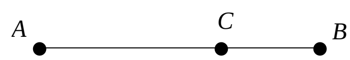
TAKE IT FURTHER

14. Create a model of this investigation using geometry software. You should be able to move point P and see where the creases would be. How can you make the "creases"?

CIRCLE INTERSECTIONS

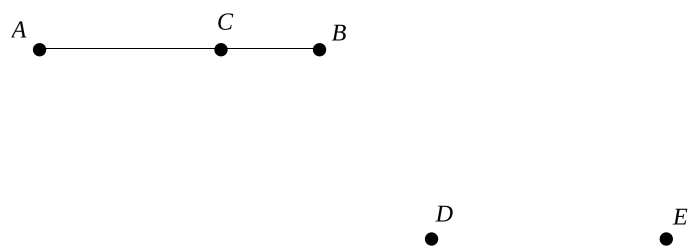
THE SETUP

Using geometry software, construct $\overline{AB}$ and place C near (but not precisely at) the middle of $\overline{AB}$. Then construct $\overline{AC}$ and $\overline{CB}$.



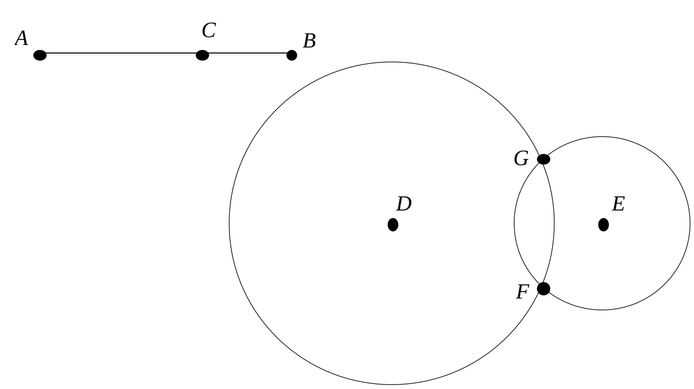
Both $\overline{AC}$ and $\overline{CB}$ have been constructed as segments on top of $\overline{AB}$.

Place points D and E so that their distance from each other is less than AB.



$DE < AB$

With D as center and AC as radius, construct a circle. With E as center and BC as radius, construct a second circle. Find the intersections of these two circles, and label your entire construction as illustrated here.



BACKGROUND CHECK

What lengths in the diagram determine the lengths of the sides of triangle *DGE*?

15. As you move C back and forth on $\overline{AB}$, notice that the circles do not *always* intersect. Use the Triangle Inequality Theorem to explain why the circles sometimes intersect and sometimes do not.

THE INVESTIGATION

Trace the intersections of these two circles and then experiment with the diagram by dragging A, B, C, D, and E. Patterns that you see as you move these points are kinds of invariants.

16. Trace the intersection points (F and G in the picture on the previous page), and describe what you see when you move A around.

a. What shapes do the points trace out as A moves?

Any aspect of the pattern that does *not* depend on the placement of *C* is an *invariant*.

b. Does the pattern you see as you drag A depend on where C started out along $\overline{AB}$? (For example, does it matter if C starts out close to A, far from A, or close to the middle of the segment?) What is the invariant here?

c. Make a reasoned argument for or against this statement: "Whatever pattern (invariant) I see when I move A, I should see the same pattern when I move B."

17. This time, leave A and B fixed, and move C along $\overline{AB}$. What pattern do you find in the intersection points of the circles?

18. Now move D or E around while leaving everything else fixed. Describe what happens.

CENTERS OF SQUARES

BACKGROUND CHECK

19. How can you find the center of a square without measuring or folding?

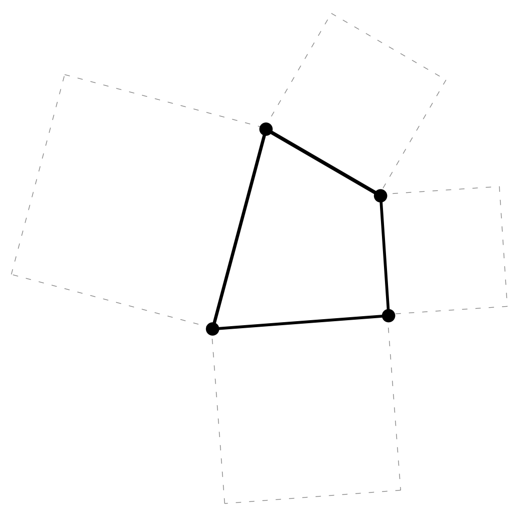
THE INVESTIGATION

Use geometry software for this investigation.

What makes a quadrilateral *arbitrary*?

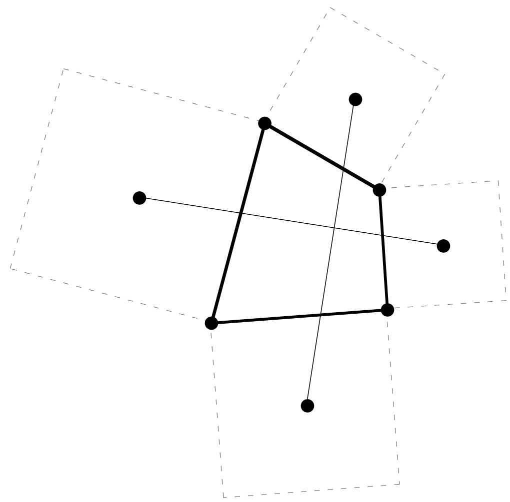
Your software may allow you to darken or color some lines, and render others as dotted lines. When a construction has many lines in it, changing how some of them look can make it easier to "read" the picture.

Make an arbitrary quadrilateral. Then, on each side, construct a square facing outward.



It's important to construct *these squares, and not just draw them to look about right!*

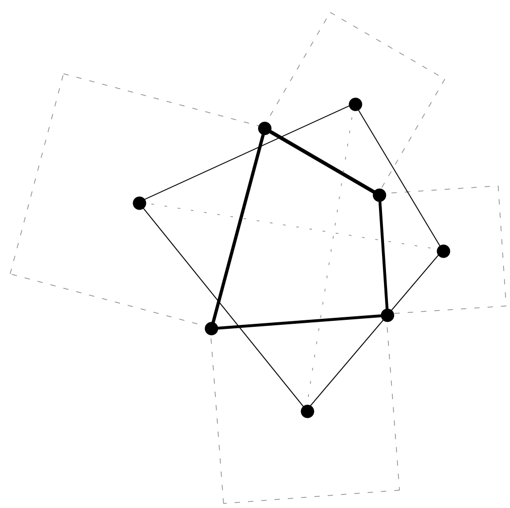
Find the center of each square, and then connect opposite centers.



20. Drag the vertices and segments of your original quadrilateral. What invariants can you find in this situation?

21. Can you find an explanation for one (or more) of your invariants based on the geometry of this construction?

Connect the four centers of squares in order to get a new quadrilateral.



22. How does the shape of this new quadrilateral depend on the original quadrilateral? If you make the original one a rectangle, what shape is the new one? Can you make the new one a trapezoid—what original shape do you need? List any conjectures you have about how the two shapes are related.

A *counterexample* is one instance that proves your conjecture is false.

23. One group of students compared the areas of the original quadrilateral and the derived quadrilateral. They conjectured that the derived quadrilateral always had the greater area. Does this appear to be true? If not, give a counterexample. If so, try to refine the conjecture even further. How much larger must the second area be?

CONSTRUCTING INVARIANTS

Here is an "invariant search" in reverse. Up until now, geometric figures were described in this module, and your job was to build them, study them, and find invariants. Here you are given an invariant and asked to construct a geometric figure which produces it. Be creative. There are many different solutions to most of the problems.

GENERAL DIRECTIONS

Use geometry software to construct a figure that has the specified invariant. Then write a set of crystal clear directions for making that figure. Be prepared to explain your solution.

THE PROBLEMS

24. Construct two circles that have an invariant 2:1 ratio of circumferences. That is, the circumference of one, even as it grows and shrinks, is always twice the circumference of the other.

25. Construct a triangle and a rectangle whose areas remain in a fixed ratio when either figure changes size.

26. Construct a square and a circle that have an invariant ratio between their areas—a ratio that remains fixed as either shape grows or shrinks.

Problems 27 through 30 are really asking you to construct constant sums (perimeters) and constant products (areas). These problems will take some clever thinking!

27. Construct a triangle whose area can change but whose perimeter is invariant.

28. Construct a triangle whose perimeter can change but whose area is invariant.

If these objects are built into your software, do some clever thinking anyway—how could you construct them with circles, parallels, perpendiculars, chords, . . . , things built from the basic tools.

29. Construct a rectangle whose area can change but whose perimeter remains fixed.

30. Construct a rectangle whose perimeter can change but whose area is invariant.

FOR DISCUSSION

Discuss the different strategies that you used to work on the invariant problems in this investigation. Explain why you chose to use a particular strategy, and describe which methods were useful and which were not.

Investigation 1.19

GUESS-AND-CHECK

Doesn't this ever stop? Just how many ways *are* there?!

There is usually more than one good way to solve a problem. Over time, you will develop more ways to think about things and better intuition about which ways are most helpful for a particular kind of problem. The investigations in this section of the module focus on three valuable thinking tools.

Guessing can be very helpful or a total waste of time: it depends on the situation, and on how you use your guesses. Problem 1 below is a good example in which thoughtful guessing can help.

The point is certainly *not* that you can just guess the right answer. In fact, you could spend the rest of your life on wild guesses without hitting the right one. But if you *check* a wrong guess to see just *how* it is wrong, you may learn how to make your next guess better.

For *some* problems, the guess-AND-CHECK strategy is the only method yet known!

For many problems, the guess-AND-CHECK strategy can be the easiest way to start. This is especially true if you have a spreadsheet, a programmable calculator, or Logo to do the calculations for you. Then you can concentrate on hunting logically for an answer.

When you try the guess-AND-CHECK strategy in the following problems, think about what *kind* of checking (and guessing) you must do in order to be efficient.

SPENDING THE MOST

You did it! You won first prize at your school's math and science fair—a $250 gift certificate to the best clothes store you know! And just at the right time, too. You've really been needing some new clothes. But there's a catch: you have to use the certificate in just one shopping spree, and on only two kinds of items. Anything left over cannot be cashed in or saved for another day. Suppose you decide that the two things you need most are jeans and shirts. Suppose you also decide that you can't afford to use any of your own money at this time. The total cost for the jeans and shirts must therefore be less than (or exactly equal to) $250.

In Massachusetts, where the authors live, there is no sales tax on clothes. You can add sales tax if your state has it, or ignore it as the authors did when solving this problem.

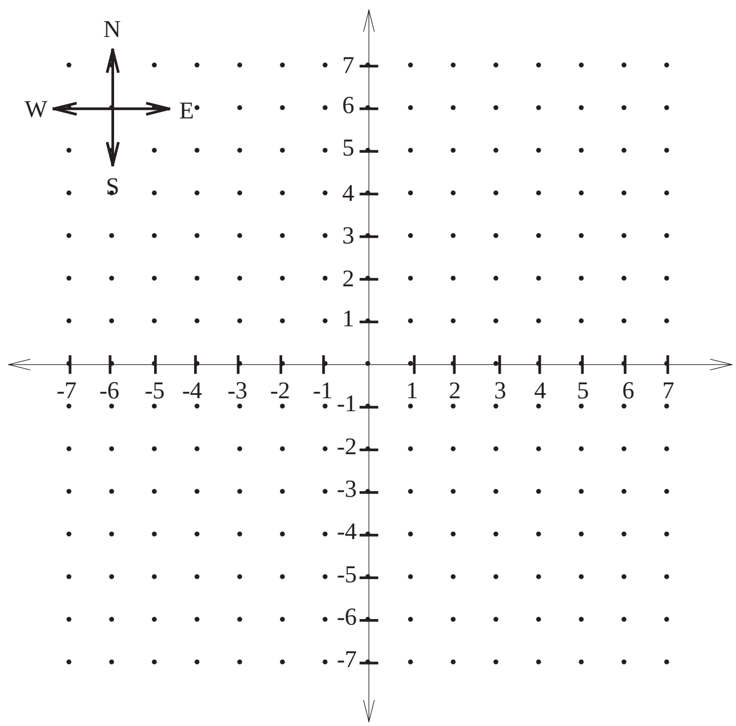
1. The jeans you like are on sale at $29.95 a pair. The shirts are $15.99 each. How many pairs of jeans and how many shirts should you buy? Your answer should include three things:

- the number of pairs of jeans and number of shirts;
- how you came up with these numbers; and

- an explanation of what makes you sure there's not a better solution (one that spends more of the $250).

FINDING YOUR HOMEWORK

Here's a kind of hide-and-seek game to play with one other person. Both of you need to make a grid like this (but big enough to show your whole classroom):



To make sure that (4, 3) and (3, 4) name different points, and so everyone can know which is which, people have long agreed to state the east/west information first, and the north/south information second.

Setup The Hider imagines a place, somewhere in the classroom, to "hide" the Seeker's math homework. The classroom is represented by the grid: Dead center of the room is named (0, 0). The location four steps "east" of center and three steps "north" is (4, 3). Similarly, the location one step "west" of center and five steps "south" is (−1, −5).

The Hider (secretly) marks a location for the homework on one of the grid's intersections, and doesn't show the grid to the Seeker. The Hider then helps the Seeker find that location by giving feedback on the Seeker's guesses.

Play After the Hider has "hidden the homework," the Seeker guesses a location in the room by naming it on the grid.

If the Seeker guesses correctly, the Hider must say so. Otherwise, the Hider must say (honestly!) whether the homework is "east," "north and east," "north," "north and west," "west," "south and west," "south," or "south and east" of the guessed location. The Seeker then guesses again, gets new feedback, and so on until the homework is found.

The word "guess" is ambiguous here. If the Seeker can *reliably* find the homework in six "guesses," that really means five guesses and one sure thing. The last "guess" is no guess at all; the Seeker *knows* where the homework is.

How Does One Win? Eventually you'll find the homework, no matter what. Winning means coming up with an *efficient* strategy, finding the homework in the fewest guesses.

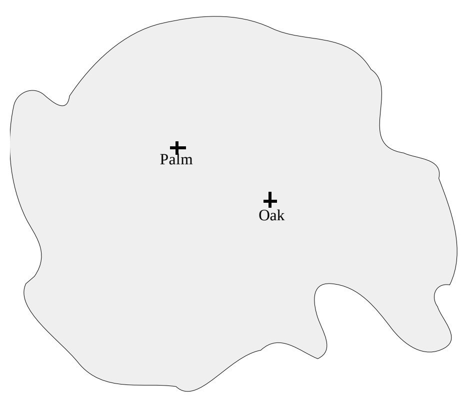
2. **Write and Reflect** Play the hide-and-seek game several times with a partner, alternating being Hider and Seeker, and test out a few guessing strategies. Can you find a strategy that reliably finds the homework in fewer than six guesses? Fewer than five? Where should your first guess be? How should you make your next guess? With your partner, write a clear description of the best strategy the two of you found.

How much information do you get from a question like: "Is the sum of the coordinates less than 5?"

3. **Changing the Game** Instead of guessing locations, you can ask the Hider any yes/no questions. (You could ask, for example, "Is the homework at (3, 4)?" For a guess like this, however, the answer won't help you much in making your next guess.) Can you still always find the homework? List four or five good questions to ask, and explain what you gain by asking them.

CAPTAIN BONNY'S TREASURE

Anne Bonny was a real pirate. In 1720 she was caught and sentenced to hang for her crimes, but she mysteriously disappeared. The story here, however, is pure fiction. The authors chose to use the name of this real pirate just because she's such an interesting character!



The Spanish king had large ships called *galleons*, many of which were used to carry to Spain treasure that was looted from the "New World."

This crude map shows the island where Captain Anne Bonny buried the treasure she stole from the Spanish galleon *Alhambra* in 1732. Except for two lone trees—a tall palm and an old wind-bent but massive oak—the small island grew only salt grasses and low bushes. Leaving her great pirate ship anchored at a distance, too far for her crew to see where she would hide the treasure, Captain Bonny set off to the island in a small sailing vessel. Instead of marking the burial site on the map, Captain Bonny wrote a description of how she picked the spot to bury her treasure.

> The only objects of any size to be found on this island were two trees, an oak and a palm, so I chose them as landmarks. I first scratched a spot in the ground where I was standing, and then counted off my paces toward the palm. Once I reached the palm, I turned to my left 90° and walked that same number of paces away from the palm. I left my shovel there and went back to my starting point.
>
> This time, I faced the oak, and counted off the number of paces it took me to reach it. From there I turned right 90° and walked the same number of paces away from the oak as I had to reach it. With my cutlass, I marked the spot where I stood. Then I paced out the distance from there to the shovel that I had left earlier, took my shovel, and paced out half the distance back to my cutlass. Then I buried the treasure.
>
> I shall erect a marker where I started. Then I shall find this treasure easily when I return, for I can pace out the same paths once again.

4. Copy the map and mark some place where the captain may have started. Then draw the two paths that she would have taken. Mark an **X** where the captain would have buried her treasure.

After she had buried the treasure and smoothed over any traces of digging, Captain Bonny marked her starting place by building a gallows from the mast and yardarm of the boat in which she had come ashore. She wanted the gallows to scare off intruders who might happen upon the island. She rowed back to her ship.

A pirate's life was often a difficult one, and Captain Bonny met with a most unfortunate fate and died before she could get back to the island. Two years later, the pirate ship's first mate and its boatswain found her description in a secret compartment in her quarters. They returned to the island for the treasure. The palm and the oak were still there, but there was no trace of the gallows. Apparently, from the look of the island, a furious hurricane had washed it away. Without knowing where to start the paces, the first mate and the boatswain gave up the treasure as lost, and cursed the Captain's devious ways.

Pirates didn't always have the best habits of mind. These two gave up too soon, but it's not yet too late for *you*! Is there any way to figure out where the treasure is buried or at least to narrow the location down to something more manageable than the whole island?

5. In Problem **??** you were asked to guess a starting location and follow Captain Bonny's directions. Compare the location of your **X** (marking the treasure) to those found by other students. How does the location of the gallows affect the location of the treasure? Is there an invariant that can help you find the treasure?

MORE PROBLEMS

It's not unusual for guess-AND-CHECK to be quicker and easier (and therefore better!) than algebraic methods for solving certain algebra word problems.

6. The product of two positive integers is 21, and their sum is 22. What are the integers?

Of course, check to make sure your solution is correct!

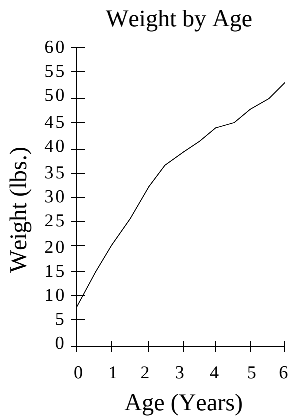
7. Here is a problem adapted from a popular algebra text. Solve the problem any way you like, and explain your method. *"A pen and pencil set costs $4.99. The pen costs $0.89 more than the pencil. How much does the pencil cost?"*

When *is* algebra a genuine help?

8. Check the word problems of one of your school's algebra books. See if you can find some problems that are more easily solved by some kind of guess-AND-CHECK strategy than by setting up a formal equation and working out the algebra.

Investigation 1.20

REASONING BY CONTINUITY



Did this person ever weigh 30 pounds?

Newborns start out at different birthweights. You now weigh much more than you did as a baby. Nonetheless, it can be said for certain that there was a time when you weighed exactly 30 pounds.

How could such a thing be known *for certain*? Well, there was a time when you weighed less than that. And now you weigh more. So, no matter how your weight may have increased or decreased in between, there must have been at least *one* time, however brief, when you weighed exactly 30 pounds.

CONTINUOUS CHANGE

1. At 6:00 one morning, the temperature in Boston was 64°F. At 2:00 that afternoon, the temperature was 86°F. Can you be certain that there was *some* time that day when the temperature was exactly 71.5°F? Can you tell what that time was? Explain your answers.

2. In the 1950s, the town of Sudbury had a population of roughly 5000. By 1990, the population was well over 40,000. Can you be certain that there was *some* date at which the population was exactly 10,000? Can you tell when that date was? Explain your answers.

3. Some car companies advertise how quickly their cars can go “from 0 to 60.” If a car goes from 0 to 60, is there a time when it’s travelling exactly 32 mph? Explain.

This problem is, in some ways, the most important problem in this entire investigation!

4. **Write and Reflect** Problems 1 and 2 look nearly the same except for the numbers, but they are profoundly different, and have very different answers. Explain why.

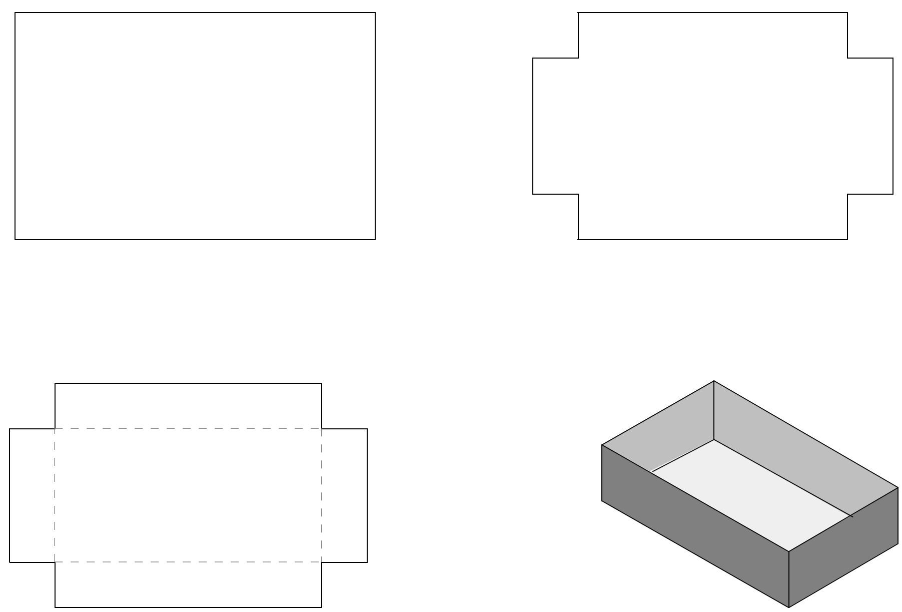
5. Some people find Problems 1 and 3 essentially the same, while others claim that they are quite different. Where do you stand? Why?

6. **Meeting Yourself on the Road** Starting at dawn, you hike up a mountain, arriving at the summit roughly at dusk. You fix a delicious dinner, camp the night, and sleep quite late the next morning. After enjoying the beautiful view and a great breakfast, you hurry down the mountain following the same path you took up (stopping briefly in the middle to catch your breath), and arrive at the bottom while the day is still bright.

   Is it *possible* that you passed some point along the path at exactly the same time of day on both ascent and descent? Is it *certain* that there is such a point? Could there be exactly two such points?

THE BOX PROBLEM

By cutting identical squares out of each corner of a $5'' \times 8''$ index card, you can create a shape that can be folded into an open box.



What is *volume*?

7. How does the box's volume change as the size of the cutouts increases? Perform some experiments to see how the volume depends on the size of the cutouts. For now, do *not* bother with formulas or numbers. Just notice whether the volume always increases, always decreases, stays the same, or does something else as the size of the cutouts increases.

8. From your experimenting and your thinking, give a reasoned answer to each of these questions.

These two questions do not ask whether you can *find* such a cutout. You are to state only whether such a cutout exists.

a. Is there at least one size of cutout that makes a box with the greatest volume? How can you be sure?

b. Is there a cutout that makes a box with the smallest volume? How can you be sure?

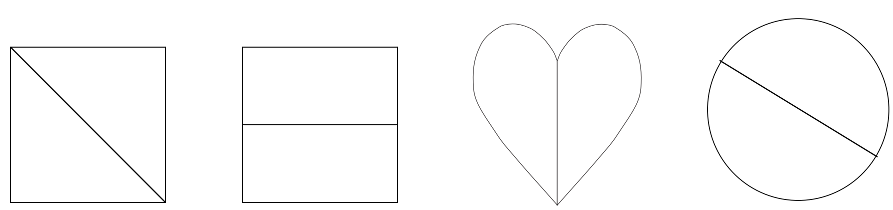
9. Make the box with the largest volume that you can. Describe what thinking led you to *that* box. What reasoning got you to change the cutouts in the ways you did?

There are several possible ways of setting up computer experiments that can help you test for the largest possible box volume.

10. Find the box in your class that has the most volume. What cuts were made for that box? Can you be certain from the class's work that no bigger box is possible?

THE "HAM SANDWICH PROBLEM"

Two Dimensions For some figures, it's easy to find a line that cuts the figure into two parts that have equal area.



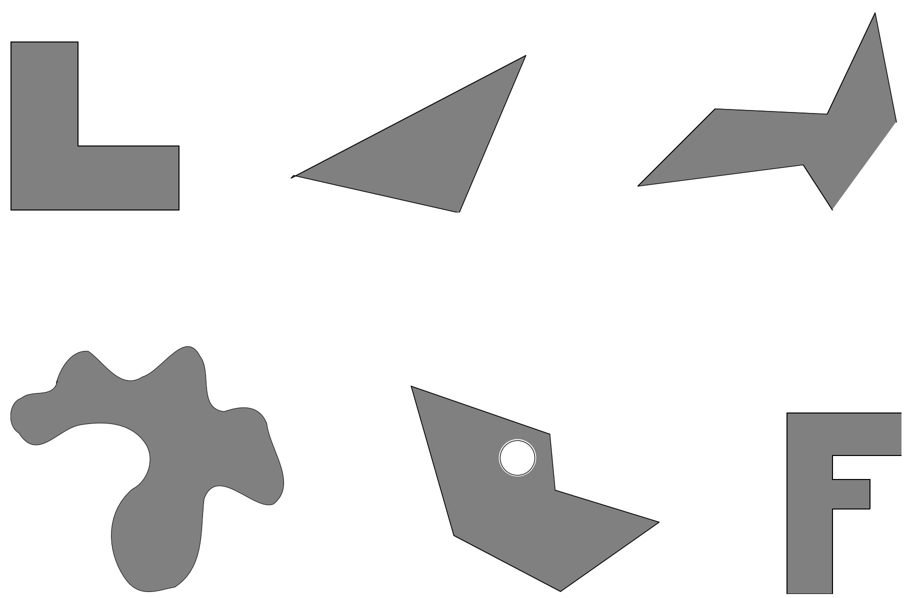
Think about what assumptions you make about these shapes, and include those assumptions in your answer to this problem.

11. Explain why each of the cuts shown on the previous page would work. That is, how do you know that the two pieces you would get have the same area?

12. For some shapes, it's not as obvious how to cut them into two pieces of equal area with a straight line. For *some* shapes, this may even seem impossible to do.

a. Try to do it for the shapes below. How could you *know* if you succeeded?

b. Perhaps for one or another of these shapes there really is no straight line that cuts it into two equal areas! How can you be sure whether there *is* such a line? For any of these, could there be *more* than one such line?



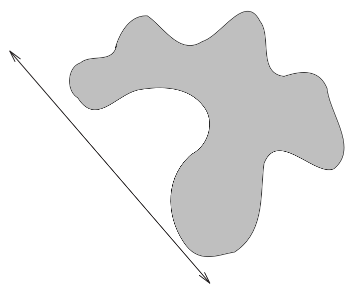
13. Try to create a shape whose area absolutely can *not* be bisected by a straight line.

Try to create a shape whose area can be bisected by only *one* particular straight line and no other.

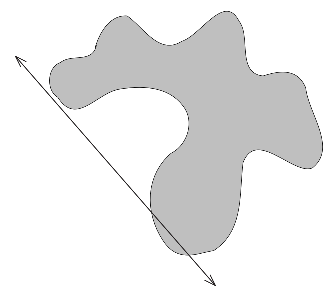
Even if it is hard to *find* a line that will cut some shapes in half, it's possible to convince yourself that such a line exists. The argument requires reasoning by continuity. Take one of the shapes in Problem 12. Copy the figure and draw a line somewhere

You can actually perform this experiment by using some string or a pencil to represent the moving line.

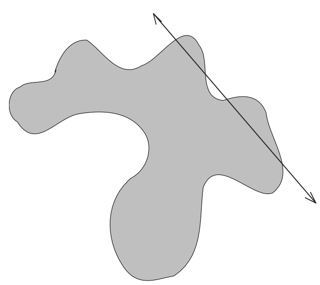
completely outside of the figure (never intersecting it). Then think of slowly passing the line over the figure, until it slides completely off again, no longer intersecting it.



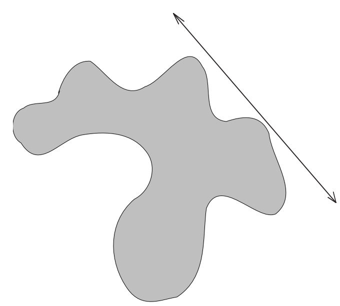
No part cut off



Tiny part cut off



Most cut off

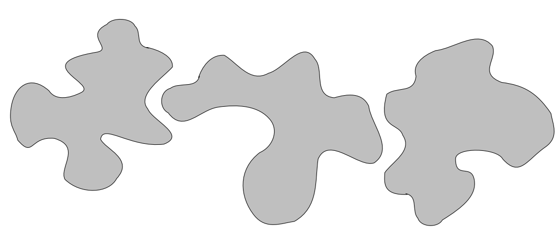


All cut off

In fact, there are infinitely many lines that work. Why?

14. Use the pictures above to write an argument that there is some line that cuts the shape exactly in half. Does knowing this help you to *find* the line?

15. Show with pictures and an explanation that it is always possible to bisect the *combined* area of three arbitrary shapes with a single straight line.



16. Show with pictures and an explanation that it is *not* always possible to find a single straight line that will simultaneously bisect the area of *each one* of three arbitrary shapes.

17. **Challenge** Is it always possible to find a single straight line that will simultaneously bisect the area of each one of *two* arbitrary shapes?

You might prefer to imagine a tomato sandwich or cheese sandwich. For the purposes of this problem, anything you like inside the sandwich will do just fine, as long as the filling consists of just one slice of one ingredient.

Three Dimensions This problem (bisecting irregular figures) has sometimes been dubbed the "ham sandwich problem" because of its 3D version, which can be stated like this:

> Imagine a ham sandwich: two slices of bread and a slice of ham. You don't know exactly how everything is arranged. For example, the ham may not be the same size and shape as the bread, and it may be off-center. Is there a plane that cuts *all three parts* of your sandwich exactly in half?

18. **Write and Reflect** Work on the ham sandwich problem. Describe your strategies. Describe any dead ends that you meet. Do you believe that the statement, "A single plane can bisect three distinct volumes" is true? Explain your reasoning.

Ways to Think About It

Remember the strategy "Change the problem"? This problem is certainly hard enough to be worth simplifying in some way.

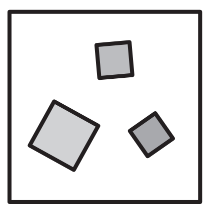
How can you simplify this problem? You could reduce the number of objects from three to two or one. You could reduce the dimensions from three to two. You could think about bisecting the *combined* volume instead of bisecting each object individually. Or you may want to think about simple, regular 3D shapes (like spheres or cubes) rather than bread and ham.

DEFINITIONS AND SYSTEMS

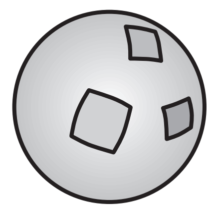
THINKING ABOUT MEANINGS: WHAT DOES "STRAIGHT" MEAN?

Topology is a field of mathematics in which one studies the properties of geometric figures that are unaffected when the figure is distorted (bent, stretched, or squashed) but not torn. The topologist observes, for example, the properties that the surfaces of a torus (donut-shape), a roll of masking tape, and a coffee mug have in common.

Jeff Weeks's book *The Shape of Space*[1] is a wonderful study of geometry and topology at both high school and college levels. The book begins with a story based on a much older book, Edwin Abbott's *Flatland*. All the characters in Weeks's story are two-dimensional creatures, and they all assumed their world was flat, a plane. All, that is, except one—A Square, by name—who had the theory that their world was actually a "hypercircle" (a circle of one higher dimension, what we would call a sphere). Now, the creatures live *in* their surface-world, not on top of a surface as we do, so they are unable to escape into the third dimension to view the surface from afar. How could these creatures check out A Square's theory?



A Square and two friends on a plane



. . . and on a hypercircle

A Square thought that a journey might settle the issue:

> "He reasoned that if he were willing to spend a month tromping eastward through the woods, he might just have a shot at coming back from the west.
>
> "He was delighted when two friends volunteered to go with him. The friends didn't believe any of A Square's theories—they just wanted to keep him out of trouble. They insisted that A Square buy up all the red thread he could find in Flatsburgh. The idea was that they would lay out a trail of red thread behind them, so that after they had travelled for a month and given up, they could then find their way back to Flatsburgh.

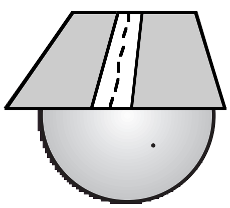
[1]Reprinted from *The Shape of Space*, Weeks, J., New York: Marcel Dekker, 1985.

"As it turned out, the thread was unnecessary. Much to A Square's delight—and the friends' relief—they returned from the west after three weeks of travel. Not that this convinced anyone of anything. His friends thought that they must have veered slightly to one side or the other, bending their route into a giant circle in the plane of Flatland." (Weeks, 1985, pp. 5, 6)

Assume there are no mountains, trees, rivers, or other obstacles.

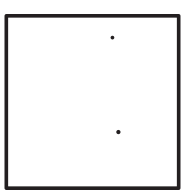
FOR DISCUSSION

- As a practical matter, how can you *know* for certain that you are travelling straight on any given surface? Just what *is* straightness?
- In *our* world, what is meant by a "straight road?" A road that is straight like a laser-beam won't stay on the earth, because the earth is round. A road of any significant length must curve through three-dimensional space in order to stay on this earth-ball of ours. What, then, distinguishes the road we call "straight" from the road we call "curved?"



This is a straight road, but not on the sphere!

What is the shortest path between the two points? Is there more than one?

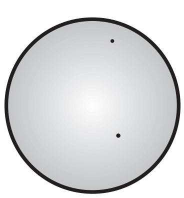


1. Ignoring details like mountains and oceans, picture two straight roads in *our* world, both starting at right angles to the equator and about a block apart, and extending (straight!) north for six thousand miles or so. Do they stay a block apart, or does the distance between them change? *Can* two roads (each at least 6000 miles long) on earth remain the same distance apart and still both be straight?

2. a. Mark two points on a plane and draw the shortest path between them. Take that path as the *definition* of "straight" on the plane. Extend that path in a natural way at both ends. What figure is that extended path?

b. Is it possible to find some arrangement of two points on the plane so that there is *more than one* shortest route from one to the other?

What is the shortest path between *these* two points? Is there more than one such path?



3. a. Now mark two points on a sphere, such as a styrofoam ball or a globe and draw the shortest path between them. (The path must be *on the sphere*, not going through the interior of the sphere.) Take that path as the *definition* of "straight" on the sphere. Extend that path in a natural way at both ends. What figure is that extended path?

b. Is it possible to find some arrangement of two points on the sphere so that there is *more than one* shortest route from one to the other?

Great circle is the name for a "line" on a sphere. (Great circles are considered the equivalent of lines because they represent "straight" paths on a sphere, or along which one can travel the shortest distance between two points.) Any great circle divides a sphere into two equal parts, or hemispheres.

The triangle's sides (as well as the square's) must be line segments, so they must be segments of great circles, not just any path between two points.

4. Not all the constructions that can be made on a plane can be made on a sphere. Can a triangle be drawn on a sphere? If you say "yes," tell how the figure you draw fits the definition of a triangle. If you say "no," explain why not.

5. What about a square? Can A Square exist on a sphere? Explain.

6. On a plane, the sum of the angles of a triangle of 180°.

a. Draw a triangle on a sphere. Measure each of the angles. What is the sum of the angles for your triangle?

b. Draw another triangle on your sphere, either much bigger or much smaller than the one you drew in problem 6a. Measure and sum the angles again.

c. Is it possible to draw a triangle with three right angles on your sphere?

7. What can you say about the sum of the angles of a triangle on a sphere? Is the sum constant, as it is in a plane? Is the sum bounded by any limits? Test several more triangles on a sphere and make a conjecture.

Life on a Sphere

Geometry on the plane makes a lot of use of parallel lines. You've just investigated a world *without* parallel lines. So what does *parallel* really mean?

When you're working on a sphere, not in the plane, the first definition should really be changed to read "Parallel lines lie on the same sphere, and never intersect."

There are two common definitions for parallel lines. You can work with whichever definition you prefer.

i. Parallel lines lie in the same plane and never intersect.

ii. Parallel lines are everywhere equidistant.

Lines that never intersect but aren't parallel have another special name: *skew* lines.

8. Why is the statement, "Parallel lines never intersect" (compare with definition i) not a good enough definition? Describe two lines that never intersect but are not parallel by the second definition.

9. You can't draw parallel lines on a sphere. *Why not*? Use your model of a sphere, and try to draw two great circles that never intersect. Write a description of what goes wrong. Include pictures!

10. There are no parallel lines on the surface of the Earth. Can there be perpendicular lines? If not, why not? And if so, how are you defining perpendicularity on a sphere?

One rule that's different is that the sum of the angles on a triangle (on a sphere) is always greater than 180°.

What you've been doing in the last few problems is exploring a new system: the geometry of a sphere. This system is different from the geometry of a plane: it starts from different assumptions and, as a result, different rules (or theorems) arise.

A good way to learn about a new system is to see how it's the same as, and how it's different from, a system you already know. You've already looked at squares, triangles, perpendicular lines, and parallel lines. Use the suggestion below to investigate other figures or relationships on your own.

Things that are true regardless of what surface is being investigated are "invariant under a change of surface."

11. **Write and Reflect** Think of some familiar properties or shapes in planar geometry. Explore related ideas or shapes on your sphere. Write a short paper to describe your spherical geometry findings, and to discuss things that are true in *both* systems.

12. **Challenge** You explored the angles in triangles on a sphere, and you probably found that the smaller the triangle, the smaller the angle sum (though it never goes below 180°). Investigate this idea. Can you find a more-precise relationship between the area of a spherical triangle and its angle sum? Write about how you investigated the idea and what you found.

TAKE IT FURTHER. . . . .

Back to our story . . . A Square was not to be defeated by doubters, and so set out on another expedition, this time to the north, laying a trail of blue thread. When he returned two weeks later from the south, *everyone* was surprised. Most Flatlanders, of course, were surprised that he got back at all, and assumed that he got lucky again and veered off course. But A Square, too, was surprised. This second journey was much too short. Even stranger, he had never crossed the red thread that had been laid out during the first journey.

13. Write and Reflect If A Square's world was a *plane*, as the Flatlanders imagined, then the first journey with the red thread must have somehow looped around to allow A Square to arrive from the west after having started out east. Experiment by drawing closed-loop paths on the plane—a red one starting east and returning from the west and a blue one starting north and returning from the south. Can you explain the shorter time of the blue path? Can you find a way for the blue path not to cross the red path before returning home?

14. Write and Reflect If A Square's world was what we'd call a *sphere*, as A Square theorized, then it is possible for the red journey to go "straight" around the globe, but it is still possible for it to have been some other kind of loop. How can you explain the shorter time of the blue path on a sphere? On a sphere, can you find a way for the blue path not to cross the red path before returning home?

15. Write and Reflect Assuming the red thread was never broken or removed, what could account for A Square's results? Think about some other shapes for worlds. What shape must the world be if it is possible for A Square to make the second trip without crossing the red thread?

BUILDING UP FROM RULES

Use wires and beads (or other materials provided by your teacher) to design *one object* that follows *all four* of these rules:

- Each pair of wires has exactly one bead in common.
- Each pair of beads has exactly one wire in common.
- Every bead is on exactly three wires.
- Every wire contains exactly three beads.

Of course, the objects were all made from the same set of four rules, so they should have *at least* four attributes or properties in common. Are there more than four?

16. Compare all of the objects built in your class. In what ways are they different? In what ways are they alike? (You might look at attributes like *shape*, or *number of beads*, or *number of wires*, or *number of places where two wires join at an endpoint*. What else might be useful to compare?)

Some of the objects built in your class may include curved wires. Recall what meaning of "straight" allowed you to regard great circles as "straight lines" while working on a sphere. Does that meaning of straight justify regarding the wires as "straight lines" in this new geometry?

Draw a picture of your object. If you call the beads "points" and the wires "lines," then you have modeled a system known as a *seven-point geometry*. You can draw conclusions about your seven-point geometry based on all the models constructed in class.

17. How many lines must you have in a seven-point geometry?

18. Triangles are still figures with three sides (the sides can be lines or line segments). Draw pictures of some of the triangles you can find in your seven-point geometry.

19. List two other conclusions you can draw or objects you can define in your seven-point geometry.

PERSPECTIVE ON FINITE GEOMETRIES

Studying finite geometries often provides insights that are harder to discover when we are working with infinitely many points. This section describes work that high school geometry classes have done with a 25-point geometry.

William Kramer, a high school math teacher, decided to investigate a finite geometry with one of his classes. He and his class began an investigation of a 25-point geometry that he has continued with every geometry class for 28 years.

Mr. Kramer's class began with just six definitions:

Points: There are exactly 25 points in the geometry, and they are given the letters $A, B, C, \ldots Y$.

Lines: There are exactly 30 lines in the geometry. Each row and each column from the blocks below is a line.

A	*B*	*C*	*D*	*E*		*A*	*I*	*L*	*T*	*W*		*A*	*X*	*Q*	*O*	*H*
F	*G*	*H*	*I*	*J*		*S*	*V*	*E*	*H*	*K*		*R*	*K*	*I*	*B*	*Y*
K	*L*	*M*	*N*	*O*		*G*	*O*	*R*	*U*	*D*		*J*	*C*	*U*	*S*	*L*
P	*Q*	*R*	*S*	*T*		*Y*	*C*	*F*	*N*	*Q*		*V*	*T*	*M*	*F*	*D*
U	*V*	*W*	*X*	*Y*		*M*	*P*	*X*	*B*	*J*		*N*	*G*	*E*	*W*	*P*

Segment: A segment consists of two points and all of the points between them on a single line.

Length: To specify the length of a segment, one must give two pieces of information: the number of steps it would take to go from one end to the other (the shortest route)

and whether the segment is part of a row or column line. So $(2, col)$ and $(2, row)$ are different lengths.

Perpendicular: Two lines are perpendicular if one is a row and the other is a column *in the same block*.

Parallel: Two lines are parallel if they are both rows or both columns *in the same block*.

From these definitions, students found midpoints of segments, perpendicular bisectors, polygons, and even circles. They found that the geometry has no rays. Mr. Kramer's students were accustomed to defining angles in terms of rays, and so they concluded that there were no angles, but they felt that they could still reason about "perpendicular" and "parallel" lines by thinking about distance and intersection. Your class may want to explore this geometry a bit, so we won't give any more away.

20. How many different segment lengths are there in this 25-point geometry?

21. How might a triangle be defined in this geometry? Are squares and rectangles possible?

TAKE IT FURTHER

22. **Write and Reflect** The wording of a definition may sometimes seem to be just a matter of style, but it can make all the difference in the world. Here are two reasonable ways one might define a midpoint of a segment.

a. "Of all the points equidistant from the two endpoints of the segment, the midpoint is the one with the *shortest* equal distance from both ends."

b. "The midpoint is the point on a segment equidistant from the two ends."

In the plane, these two definitions are equivalent. But on the sphere, they are not: each definition identifies a unique point, but the point selected by one definition may not be the same as the point selected by the other. And in the 25-point geometry, only one of those definitions guarantees that every segment even *has* a midpoint.

Show why these are equivalent definitions on the plane but not on the sphere or in the 25-point geometry. Explain the difference carefully. In the two geometries in which the choice of definition matters, which definition do you find more useful?

Investigation 1.22

NON-EUCLIDEAN GEOMETRIES

This is a reading investigation. Read the history that follows and answer the questions at the end.

When Euclid wrote his books *The Elements*, he wanted to formalize into a system everything that he and his colleagues knew about mathematics at that time. His goal was to start with rules that were simple, believable, and few in number. To build up his geometric system, he chose the following five starting rules, or *postulates*.

1. You can place a straight line through any two points.
2. A straight line segment can be continued into an infinite straight line.
3. If you have a point (a center) and a distance (a radius), you can create a circle.
4. All right angles are equal.
5. If you have a line l and a point P not on the line, there is exactly one line in the same plane as P and l that passes through P and is parallel to l.

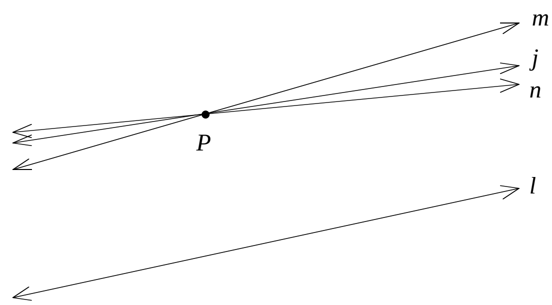
This may seem like a silly postulate if you have defined a right angle as one that measures 90°. Euclid, however, defined it this way: "When a straight line, set upon another straight line, makes adjacent angles that are equal, then each of the angles is *right*.

In addition to these, Euclid had five "common notions," such as "Things which are equal to the same thing are equal to one another." His idea was to build up all of geometry from just these few rules. Euclid (and many other mathematicians) were disturbed by Postulate 5. The fifth one didn't seem as obvious as the others; it seemed like something they should be able to *deduce* rather than assume. But Euclid couldn't find a way to do it, so he included the so-called Parallel Postulate as one of his essential few. The controversy over whether the fifth one was independent from the first four led to many interesting developments in mathematics.

After several centuries, during which quite a number of mathematicians tried to prove that the Parallel Postulate was a *consequence* of the other four postulates, someone named Girolamo Saccheri had an idea. He decided there were three possibilities: Either Euclid was right and there was exactly one line through a given point and parallel to a given line, or there was no such line, or there were more than one. If he could prove the other two cases were wrong, then the first one must be right. His strategy was to choose one of the possibilities and assume it to be true. Then he would show how the possibility allowed him to prove two consequences that contradicted each other.

Saccheri didn't have much luck with his methods. If he assumed that there were no parallel lines, he got contradictions. He knew, then, that a geometry somewhat like Euclid's (one in which Postulates 1–4 were satisfied) but without any parallel lines

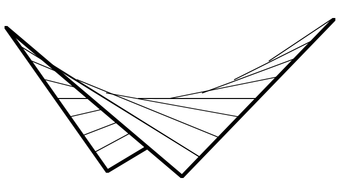
was impossible. He got lots of crazy results by assuming he could have more than one parallel line, but he didn't get any results that contradicted each other.



If m and n are both parallel to l, then so is j.

For example, in the picture above, if lines m and n are both parallel to l, then they never intersect l. If you assume there are two such lines through a point that are both parallel to a given line, then you can prove that there are infinitely many such lines. Intuitively, since line j—and every other line that's between m and n—stays between the two parallels, it can never intersect l either. Thus, j is another parallel.

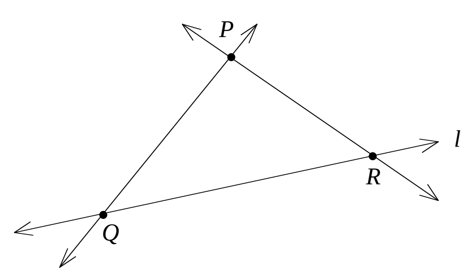
The mathematicians who were most successful in exploring the Parallel Postulate discovered the geometry of curved surfaces, such as a hyperbolic plane. To research the properties of a hyperbolic surface, check the book *The Shape of Space* by Jeff Weeks.



A segment of a hyperbolic plane

It's said that his lack of results in this line of research upset Saccheri so much that he gave up his efforts to find a contradiction. Three other mathematicians took his line of reasoning and continued it. Rather than looking for contradictions, however, they were looking for a *new system of geometry*, a non-Euclidean geometry. This geometry has come to be known as *hyperbolic geometry*, and it is one kind of geometry that can be understood on a curved surface.

When the work on hyperbolic geometry was accepted, mathematicians began to wonder if there were other non-Euclidean geometries. Georg Riemann decided to investigate a geometry in which the second postulate of Euclid was not true—a geometry without infinite lines. He reasoned that these lines would have to be more like circles than line segments—no beginning and no end but finite in length.



Because the picture on the previous page is drawn on a plane, it looks like there should be some line through *P* that is parallel to *l*. In fact, because lines are finite in this other geometry of Riemann, he could show that all lines through *P* must intersect *l* at some point! So in Riemann's geometry, neither the second nor the fifth postulate of Euclid holds.

It turns out that this geometry is tremendously useful in life on Earth—it is the geometry of the surface of a sphere, which you studied in Investigation 1.21. A line is a "great circle"—a circle whose center is the same as the center of the sphere. A line is not never-ending but finite in length, so a "straight path" on the sphere is an arc of a great circle. The shortest distance between two points on a sphere is the shortest such curve.

All great circles intersect each other, so there are no such things as "parallel lines" on a sphere.

It's amazing that, despite the common knowledge that humans live on a sphere-like surface, no mathematicians thought to study this geometry until the middle 1800s! Everyone was so convinced that Euclidean geometry was the only possiblity that they never stopped to consider that it didn't describe our world fully.

1. **Write and Reflect** Euclidean geometry was built up from physical observations—measurements of people's surroundings. Why did these measurements lead to Euclidean geometry (geometry of a plane) instead of Riemannean geometry (geometry of a sphere)?

2. Because the shortest path between two points on a sphere lies along an arc of a great circle, it would make sense for airplanes to travel on these arcs as routes from one city to another. Can you think of any reasons why a plane might not take the shortest distance between two points on the surface of the earth?

3. **Project** In the late 20th century, mathematicians have begun exploring another non-Euclidean geometry: fractal geometry. Find a book on fractal geometry, and prepare a paper, poster, or class presentation about the topic.

Investigation 1.23

VISUALIZATION EXERCISES

FROM CONJECTURE TO PROOF

In casual talk, "argument" sounds like "fight." Historically, it was quite different. The old Indo-European root *arg-* meant "to shine." (That root became a Latin root meaning "brilliance" or "clarity," which evolved into our word *argue*.) *To argue* means to make clear.

Trillions of examples don't prove a statement, but *one* measly counterexample is enough to *disprove* a statement!

What is a *prime number*?

The essence of logical reasoning, and therefore of mathematics, is moving beyond belief to *proof*. A few examples may suggest a conjecture, and a few more may get you to believe it, but even a million examples don't make a proof. Proof requires a logical, reasoned argument that ties mathematical statements together. A proof says, "If *these* (trusted, familiar) statements are true, then this new one (the conjecture) *must* also be true."

Why bother with proof if you have seen so many confirming examples that you are already completely convinced?

First of all, if you want to build new ideas out of old ones—which is what mathematics does constantly—you have to be sure the old ones are "secure" and never "fall apart." However good they look, if they are *not* reliable, then all the ideas built upon them, and then upon their results, and so on, will also be unreliable. One must be careful because there are statements that are not true for *all* cases, but *are* true for trillions and trillions of cases that one might check.

Below is a famous example of a conjecture about a prime number "generator."

Here is a formula for producing prime numbers: Take any integer, square it, add the original number, and add 41. The result is always a prime number. For example,

n	$n^2 + n + 41$
1	43
2	47
3	53
4	61
5	71

FOR DISCUSSION

Try out some more values of n. The conjecture seems pretty convincing, right? Well, in fact, it's not true for all values of n. Find a counterexample.

WAYS TO THINK ABOUT IT

One strategy is to try random values of n to see what you get. You could get lucky quickly, or you could spend a great deal of time casting about. Another strategy is to look for a value of n that might give a special appearance to the expression $n^2 + n + 41$.

There are at least two other reasons why proof is important in mathematics: First, proof often helps you understand *why* a statement is true, and helps give you new ideas. Secondly, sometimes experiments are impossible to conduct: when there is an infinite number of cases, you cannot check all of them individually! You need a way of showing that the statement is true even for cases that you have not checked or cannot check.

In the investigations in this section of the module, a number of questions will come up:

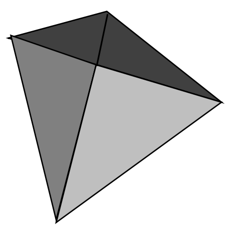
Do people "discover" new theorems or *invent* them? That question is centuries old, yet still hotly debated in the philosophy of science, psychology, and education.

- What does it take to convince you of something?
- Once you are convinced that something *is* true, what do you gain from knowing *why* it's true?
- How do people come up with proofs?
- How do people discover new theorems?

You will also meet some facts about and investigate some relationships in quadrilaterals.

VISUALIZATION PRACTICE

Before drawing images on paper, it can often help to picture them clearly in your head. Do the first two exercises in your head, without making any sketches.



A *tetrahedron* is a three-dimensional shape made from 4 triangular faces.

1. Imagine that in your school gymnasium, you place a dot halfway along each wall, waist high. You start at one dot and run in a straight line from it to the next, and the next, and so on, until you return to your starting dot. Describe the path you run.

2. Picture a tetrahedron with a point in the very middle of each triangular face. Connect each of the four middle points to every other one. Describe the resulting figure.

3. Now go back and draw what you've pictured and described in Problems 1 and 2.

Investigation 1.24

MIDPOINTS IN QUADRILATERALS

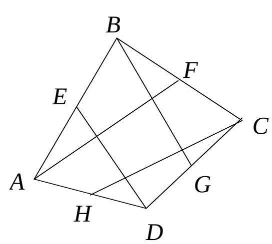
A theorem is a statement that is "demonstrably true," a statement that can be demonstrated. You probably expect "theorem" to be related to "theory." It is. It is also related to "theater"!

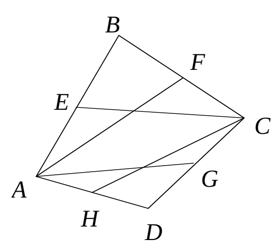
How do mathematicians find new theorems? Sometimes they set out on a specific task, looking for an answer to a question (like "How do you find the area of a circle?"). Other times, they notice something strange in an experiment, and their attempts to explain the strange phenomenon lead to a new result. The best of these "accidental" theorems lead to more general questions and, of course, new theorems.

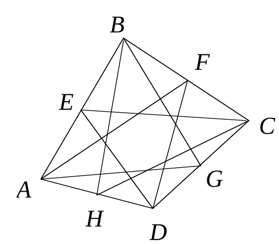
THE EXPERIMENT

This is best done with geometry software. You can start with any quadrilateral and any labeling that you like. Yours doesn't have to look like the one shown here.

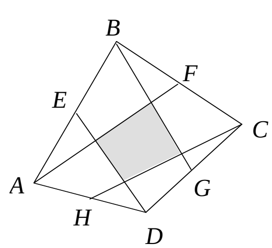
Draw a quadrilateral, and then construct the midpoint of each side. There are many ways that you could use these midpoints to create a new figure. Here are three:

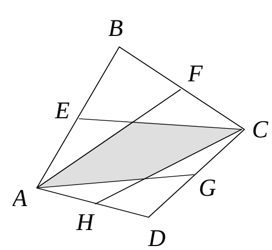


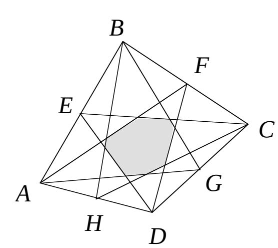




In each construction, the connecting lines surround a new figure, which is shaded in the pictures below.







1. For each construction, describe clearly how the innermost (shaded) figure was constructed. Be precise enough to allow someone who has not seen the construction to understand what you did.

2. Pick one of the constructions on the previous page. Investigate the relationship between the innermost (shaded) figure and the outermost quadrilateral. Drag the vertices of *ABCD* around and see what appears to remain invariant.

WAYS TO THINK ABOUT IT

The goal is to find invariants. But, with so much to look at, where does one start? No rule works for all problems, but some strategies help you keep track of the possibilities:

Strategy 1—Focus your attention on only the inside figure and see what doesn't change about it as ABCD changes. Possible questions: Does the inner shape always have the same number of sides? Can it be made square (or trapezoidal, or . . .)? Are some shapes forever ruled out?

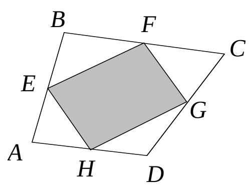
Strategy 2—Compare the inner figure to the outer one and look for invariant relationships between them. Possible questions: How does the inner figure's area or perimeter depend on the corresponding feature of the outer figure? Can the inner figure's area ever be more than half that of the outer figure?

3. Write and Reflect Describe your search for invariants. Include your observations and conjectures, and be sure to describe how you tested your conjectures. Try to be clear enough so that someone who is not in your class can understand your report.

A GEOMETRIC INVARIANT: MOVING FROM CONJECTURE TO PROOF

It's unlikely that you've had a chance to investigate your construction fully (and you might, later, choose to go back to this investigation and take it further). But, for now, the issues are *how to test a conjecture* and *how to get from a conjecture to a proof.*

Here is a new figure involving midpoints, and a conjecture to go along with it.



> If you connect the midpoints of a quadrilateral's sides in order, the result will be a parallelogram, no matter what kind of quadrilateral you started with.

Properties would be useful if they are easy to check with the tools you are using, and if a group of them are true only for quadrilaterals.

Remember, one counterexample would dismiss the conjecture immediately, saving you the trouble of trying to prove it!

For Discussion

- What are the properties of parallelograms? List as many as you can.
- Which of these properties are useful in determining whether a quadrilateral is also a parallelogram?

4. Perform some experiment, on paper or on computer, to test the above conjecture.

5. What kind of evidence would convince you that you would get a parallelogram for *any* quadrilateral? See if you can find that evidence. Describe what you have done.

6. *Why* should the inside shape always be a parallelogram?

Ways to Think About It

Some people start this kind of investigation by posing questions about special cases. For example, what if the outside quadrilateral is a rectangle? Or, what does it take to make the inside quadrilateral a rectangle? Other people just jump in and look at general quadrilaterals.

A strategy that often helps is to look for *other invariants* to explain the one that you've already found. The suspected invariant you're investigating is that the inside shape is always a parallelogram. How many other things can you find that do *not* change in this construction as you drag a vertex of *ABCD*? You can look for "fancy" things like invariant ratios (if you find them), but don't ignore the really simple things like points that don't move.

Investigation 1.25

What Do You Find Convincing?

There are many ways to convince people of facts. Here are two presentations that have shown up in classes.

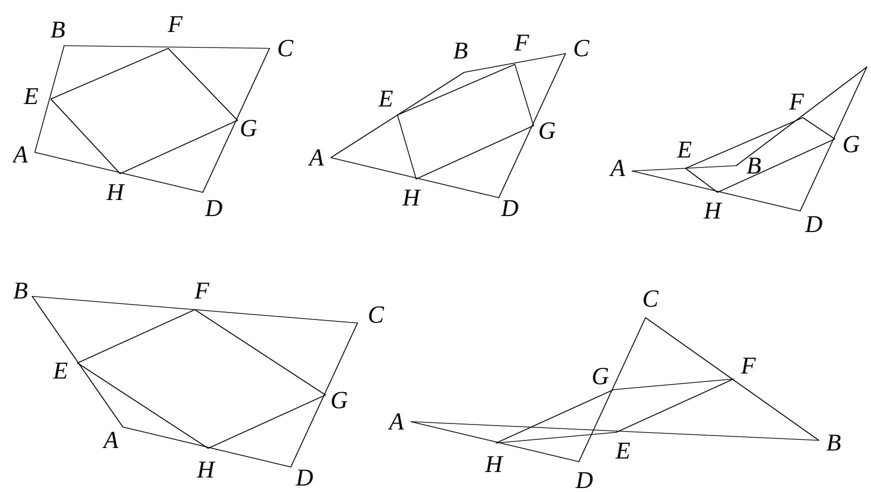
Raphe's Explanation

Raphe: Look, here's the figure we made.

He sets up the situation in geometry software.

I can move *B* around, and no matter where I move it, I get a parallelogram. I can even drag it some pretty strange places.

He demonstrates, showing pictures like these.



Raphe: It doesn't matter which of the four points I drag around; it still looks like a parallelogram. I can even drag sides around, and the inside figure still looks good.

He picks up side $\overline{AB}$ and drags it around to demonstrate.

1. Critique this explanation; that is, tell why it does or does not convince you of the following.

   - The inside figure in all of Raphe's pictures *is* a parallelogram (and not merely a close approximation of one).
   - The inside figure is *always* a parallelogram (and not merely in these particular cases).

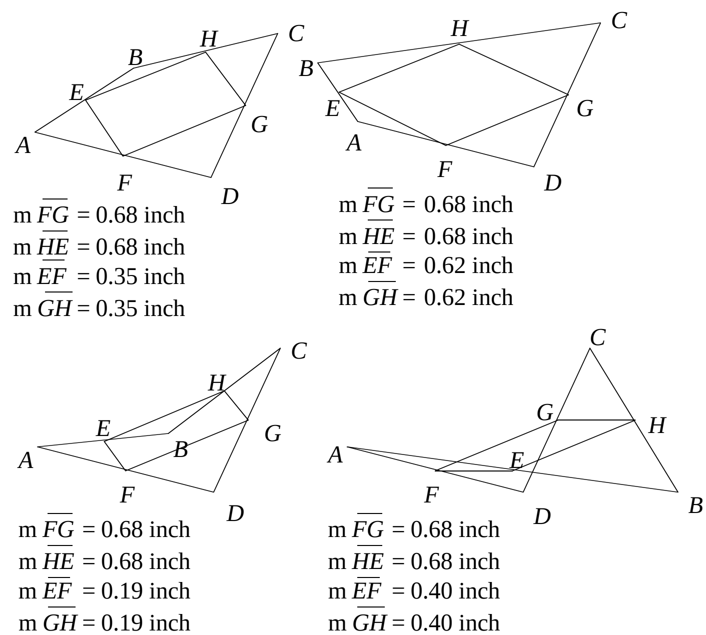
   To be fair about it, set up the experiment and actually move things around the way Raphe might have done in class.

LIZA'S EXPLANATION

Liza sets up a more elaborate demonstration using the geometry software.

Liza: Look, no matter where I drag B, I get a parallelogram because the opposite sides have the same length.

She demonstrates, showing pictures like these.



Liza: See, I can drag B around, and the opposite sides stay the same length. Even if I drag *sides* around, the opposite sides of the inside shape are always the same length. That makes it a parallelogram.

Why does one pair of sides stay at 0.68? Is the machine stuck?

2. As before, critique this explanation. Explain why it is convincing or why it is not.

- The inside figure that you've seen in all the cases Liza has shown *is* a parallelogram (and not merely a close approximation of one).
- The inside figure is *always* a parallelogram (and not merely in these particular cases).

To be fair about it, set up the experiment and actually move things around the way Liza might have done in class.

Hmmm . . . Are there any *other* segments (or distances) that aren't affected by moving *B* around?

Liza's experiment showed *two* things. First, she observed that "opposite sides of the inside shape are always the same length." To be more careful, she might have said "always *appear to be* the same length," but no matter. Secondly, the length of segments $\overline{FG}$ and $\overline{HE}$ didn't appear to change *at all* as Liza moved B around. (In all these pictures, the length remained 0.68 inch.) Such unexpected invariants in the midst of a changing system often hold the key to understanding that system.

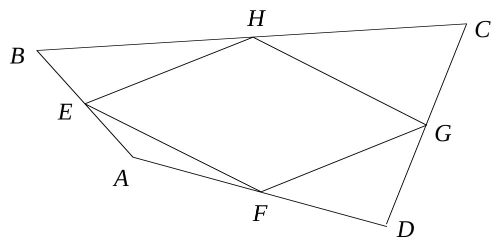
3. Look for other things that don't change when B is moved. What invariants do you find? Investigate these invariants.

4. What other measures could have been used in this demonstration? Could you get by with *fewer* than four measures?

FOR DISCUSSION

Suppose you did the same demonstration with a different piece of software and, at some point, saw the screen on the next page. Would that make the demonstration less convincing? Would you doubt the first

piece of software? The new software? The conjecture? How often would you have to see something like this before you lost faith in the conjecture?



Length ($\overline{EH}$) = 2.60
Length ($\overline{FG}$) = 2.62

Length ($\overline{EF}$) = 2.40
Length ($\overline{GH}$) = 2.40

Investigation 1.26

FINDING OTHER INVARIANTS

Liza and even Raphe gave good *data* to suggest that something special is going on. They've been convincing enough that the inside figure *is* always a parallelogram, but their explanations don't say as to *why* that should be true.

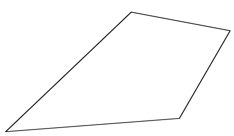
The following problems may help you think of ways to explain what Raphe and Liza saw.

You'll probably have to do some investigating on your own. Geometry software can be quite useful in these experiments.

1. One side of a triangle has length 12. How long is the segment that joins the midpoints of the other two sides? What if the one side had length 10? 18? 19?

2. State a conjecture about the length of the segment that joins the midpoints of two sides of a triangle.

A *kite* is a quadrilateral with two pairs of congruent adjacent sides.



3. Suppose a kite has diagonals with lengths 5 and 8, and a quadrilateral is formed by joining the midpoints of the kite's sides. Give the perimeter of that inner quadrilateral and describe its angles.

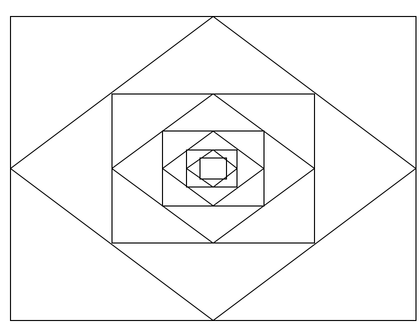
4. The diagonals of a quadrilateral measure 12 and 8. What is the perimeter of the inner quadrilateral you get by joining the midpoints of the sides of the quadrilateral?

5. The picture below began with a big rectangle whose sides measured 24 and 32. The midpoints of the sides were connected over and over again to make the design.

The Pythagorean Theorem states that the square of the length of the hypotenuse in a right triangle is equal to the sum of the squares of the lengths of the legs. The diagonal of a rectangle is the hypotenuse of a right triangle whose legs are two adjacent sides of the rectangle, so the diagonal of this outer rectangle is $\sqrt{24^2 + 32^2} = 40$.

a. List as many facts as you can about this figure.

b. What is the perimeter of the smallest quadrilateral in the figure? How did you arrive at that answer? (Did you, for example, make use of things you knew about the perimeters of the intermediate quadrilaterals?)



Investigation 1.27

MAKING THE RIGHT CONNECTIONS

Being convinced that something is true is quite different from having a clear and reasonable explanation for why it "works." Here's a scene drawn from one classroom.

Anisha's conjecture was quite shrewd, and could be proved as a theorem, but she didn't know that yet. It actually doesn't matter whether or not she knew. Her reasoning is just as sound either way.

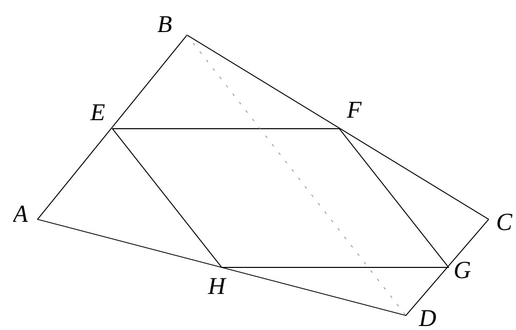
Anisha stated this conjecture:

> A line that joins the midpoints of two sides of a triangle is half as long as the third side.

She knew she had no *proof* for her conjecture, but she saw how, *if* it were true, it could explain the parallelogram result.

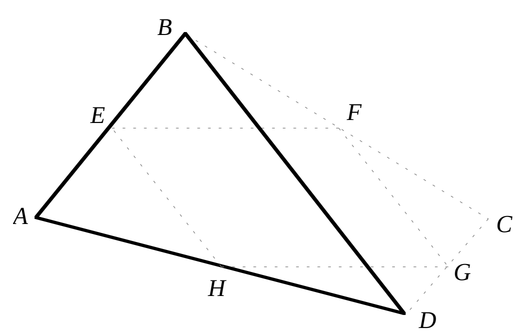
Anisha: Suppose for a minute that my conjecture *is* true. Then I can explain why the inside thing *must* be a parallelogram. First, I draw in diagonal $\overline{BD}$.

Anisha sketches this picture on the chalkboard.



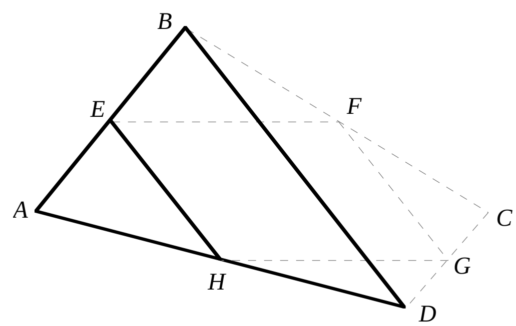
Anisha: Just look at the triangle *ABD*, and let the rest of the figure fade into the background.

Anisha darkens the three segments, $\overline{AB}$, $\overline{BD}$, and $\overline{AD}$.



Anisha: In $\triangle ABD$, $\overline{EH}$ joins the midpoints of $\overline{AB}$ and $\overline{AD}$, so, by my conjecture, $EH = \frac{1}{2}BD$.

Anisha darkens $\overline{EH}$.



Now look at $\triangle CBD$. In *it*, $\overline{FG}$ joins the midpoints of $\overline{CB}$ and $\overline{CD}$, so $FG = \frac{1}{2}BD$. Because $\overline{FG}$ and $\overline{EH}$ are both half as long as $\overline{BD}$, they are the same length. So, $EHGF$ has one pair of opposite sides that are the same length.

If I had drawn diagonal $\overline{AC}$ instead of $\overline{BD}$, I could have used the same argument to show that $EF = HG$. So the opposite sides of $EFGH$ are the same length, and *that* makes it a parallelogram. So there.

Dale: Before the explanation, I wanted to be convinced that if you connect the midpoints of a quadrilateral, you get a parallelogram. Now Anisha says that she can explain it if I believe her about this triangle result. But why should I believe that? It seems to me that we've just doubled the number of things we don't know.

Barbara: Anisha isn't saying that the parallelogram result is true. She's just saying that *if* you buy her triangle result, then you *must* buy the parallelogram result. Now, maybe we can find a convincing reason for her triangle thing to be true. If we can, great! If we can't, it's back to the drawing board. That's what math's all about.

1. Write and Reflect Choose either Dale's or Barbara's point of view and defend it in writing. Why do you agree?

WAYS TO THINK ABOUT IT

2. How might Anisha have come to pick *that* particular idea to use? That is, without having known it in advance (or being given a hint by a book

or a wizard), what parts of your investigation about quadrilaterals might lead you to stumble onto Anisha's theorem?

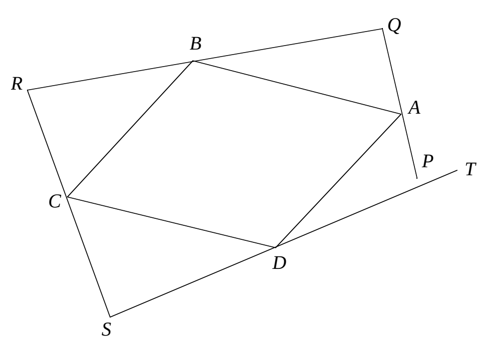
There is no rule that says when to use some idea or another. But in many cases—for example, in *this* case—invariants suggest ideas that are likely to come in handy.

Anisha's explanation depended on her definition of parallelogram. It worked because she knew that parallelograms are quadrilaterals whose opposite sides are congruent. But, suppose that all she knew was that parallelograms are quadrilaterals whose opposite sides are *parallel*. Then Anisha's conjecture would *not* have been enough to explain the parallelogram.

3. Why is Anisha's conjecture not enough in this case? What other conjecture about triangles (similar to Anisha's) *would* help? Is your answer something that you knew before?

TAKE IT FURTHER

4. Imagine starting with the inside figure and working outward. Start with a quadrilateral $ABCD$ and an arbitrary point P outside of it. From P, build $\overline{PQ}$ with A as its midpoint. Then, from Q, build $\overline{QR}$ with B as its midpoint, and so on through C and D. What special properties must quadrilateral $ABCD$ have to allow the outside figure to close (to be a quadrilateral) and not remain open as in the picture below? Explain your observations.



Investigation 1.28

Can You Say More?

Anisha's conjecture is in fact true. It's often called the "Midline Theorem," and here is its full statement:

THEOREM 1 ***The Midline Theorem***

A line that joins the midpoints of two sides of a triangle is parallel to the third side and half as long.

In the previous section, you saw how the Midline Theorem implies the big result about quadrilaterals:

THEOREM 2 ***Quadrilateral Midlines***

If you connect the midpoints of a quadrilateral in order, you get a parallelogram.

The drive to understand the world around us is very powerful. Necessity may be the mother of invention, but curiosity and imagination are the parents of knowledge!

What does knowing the proof buy you? Well, in part, people just seem to *like* understanding the *why* behind things. But also, understanding the proof of a fact can lead to new facts. The following problems show how the search for proof can lead to new ideas.

A *rhombus*: all four sides are congruent (like a square), but no special angles are required.

1. What kind of a parallelogram do you get when you connect the midpoints of a kite? Feel free to experiment.

2. Draw a few quadrilaterals that have the property that when you connect the midpoints of the sides, you get a rhombus. Is there some way to tell whether a particular quadrilateral will generate a rhombus in the middle without actually testing for the rhombus? Explain.

PERSPECTIVE ON STUDENT MATHEMATICIANS

What happens when a group of high school geometry students spends a semester playing with geometry software? They find brand new theorems that no one knew before!

The Geometric Supposer© is published by Sunburst.

In 1987, Richard Houde was teaching geometry at Weston High School in Weston, Mass. During the year, his class spent a lot of time working with geometry software called The Geometric Supposer©.

In June, the final exam involved working on the computer for one hour. The students were to examine the midlines of triangles and write up everything they found about midlines. The students made conjectures, tested them on several cases, and revised them, just as you have done in this section of the module.

By the end of the hour, Mr. Houde's class had discovered every known theorem about the midlines of triangles, and the students made some additional conjectures which were later proved but had never been known before.

Investigation 1.29

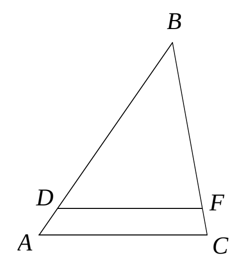
THE MIDLINE THEOREM

Why is the Midline Theorem true? That's a whole other question, one that deserves a whole other investigation. You'll find it in the module *The Cutting Edge*.

For now, here's an idea that might get you thinking. In Investigation 1.20, you learned about reasoning by continuity. You will be using that reasoning here.

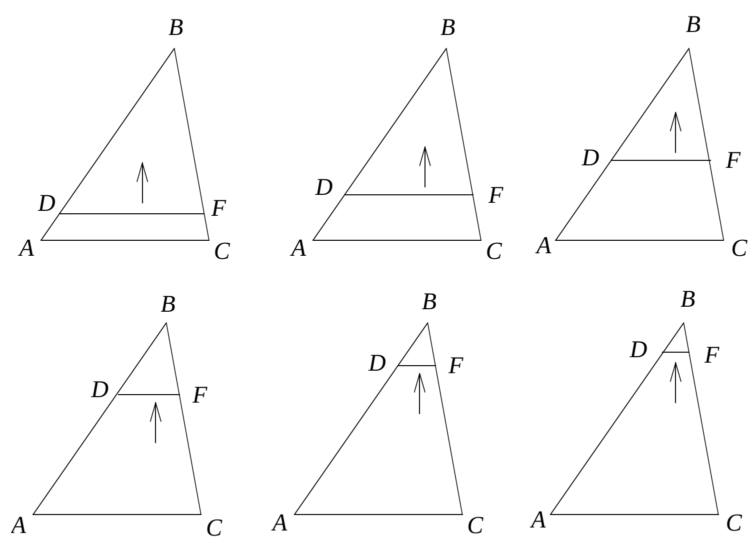
Suppose you start with $\triangle ABC$ and segment $\overline{DF}$ parallel to $\overline{AC}$ and very close to it.

Then $\overline{DF}$ is almost as long as $\overline{AC}$ (and, the closer $\overline{DF}$ gets to $\overline{AC}$, the closer DF will be to AC). So, if $AC = 10$, you can make DF as close as you want to 10.



Now imagine that you move $\overline{DF}$ up, keeping it parallel to $\overline{AC}$. Better still, build the experiment and do it.

Here are a few frames in the experiment.



So here's the argument: If $\overline{DF}$ can also become as short as you want, as it goes from 10 to 0 as you move up, there must be *somewhere* where it is exactly 5.

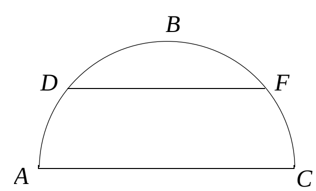
1. a. Is that so? Critique this argument. How can you be *sure* that $\overline{DF}$ will be 5 someplace?

b. Accept or reject this statement and defend your position: This "someplace" must be exactly halfway up.

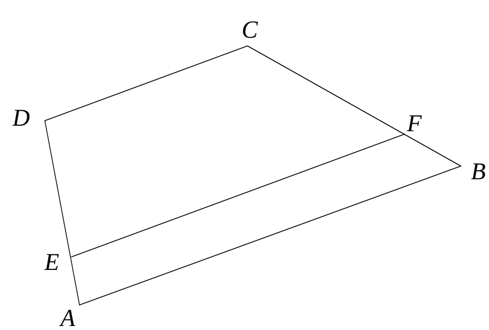
2. Draw a graph that shows how $\frac{DF}{AC}$ changes as $\overline{DF}$ moves.

3. a. Find a way of drawing the same kind of graph for a semicircle: a graph that shows how $\frac{DF}{AC}$ changes as a function of $\overline{DF}$'s position.

b. Is it true in this case that moving halfway up makes $\overline{DF}$ half as long as $\overline{AC}$? Explain your reasoning.



4. Play the same game as in Problem **??**, except slide your segment inside a trapezoid instead of inside a triangle. What is the biggest value of *EF*? The smallest? When is it halfway between the biggest and the smallest?



MODULE OVERVIEW AND PLANNING GUIDES

ABOUT THE MODULE

The beginning of a course is a critical time. Students decide what is really wanted of them—what really "counts"—in the first few days.

We believe that what counts *most* is students' thinking, so this module begins not with definitions and review, but with problems. The problems require no special background or prior knowledge, but they do require thought and persistence.

Before students can be expected to reason deeply about the properties of geometric objects, they must learn to "see" these objects geometrically. That is, they must be able to dissect them visually, to reproduce them in sketches, and to describe the parts and their relationships clearly and precisely.

This module is designed to introduce important mathematical ways of thinking through the study of geometry and assumes no prior geometric knowledge. Using a problem-based approach, it introduces essential vocabulary, ideas, tools, and drawing techniques that students will need throughout their mathematical learning. Students also encounter many geometric facts and concepts in this module that they will develop more fully later.

Similarity is one idea introduced informally here: Students draw triangles specified by angle, discover that their triangles are not identical, yet certain ratios are invariant. The emphasis for students is on looking for the invariant and proposing explanations.

Beyond developing language, background knowledge, and skills, this module helps to build a classroom culture of puzzling over problems, asking questions, starting to figure them out, and drawing tentative conclusions for later investigation.

Moreover, within the scope of just this introductory module, students *prove* over 20 theorems derived from their own investigations. In contrast to more familiar approaches in which the theorems appear in the student text, virtually all of the conjectures and proofs that give rise to theorems in this module are contained in students' responses to problems. The definitions and theorems that one usually expects the authors to have written in a student text appear primarily in our *Solution and Problem Solving Resource*. In a sense, the students are writing the textbook!

This module is ideal for:

- beginning a year-long course in geometry
- infusing serious geometry into middle-school mathematics courses
- replacing the "geometry chapter" in standard Algebra 1 texts
- beginning a 3- or 4-year integrated mathematics sequence
- adding a focus on the skills and habits of mathematical research into *any* high-school (or late middle-school) mathematics course, via investigations based in geometry

MAIN TIMELINE PLANNING CHART

The problems here are rich and the learning about content and method is extensive; many classes spend half a year in this module alone. Completing all of the main investigations and some of the extensions would take an average class 17–19 weeks.

Investigation	Description	Key Content	Pacing
1.1 Problem Solving in Geometry	This investigation gets students right into solving geometry problems. In the process, students will learn some properties of polygons, angles, and circles; a lot of geometric vocabulary; about nets and cross sections of three-dimensional shapes; and the triangle inequality.	• problems! • geometric vocabulary	7 days
1.2 Picturing and Drawing	This investigation is designed to help students create mental images well enough to analyze parts of images and to analyze visual scenes well enough to draw them. The specific drawings we ask students to create—3D letters, Escher-like impossible figures, and faces—were chosen to appeal to a broad variety of students. Skills include recognizing proportions, recognizing and representing parallels, and distinguishing between what is seen and what is imagined.	• visualization • analyzing drawings	6 days

Investigation	Description	Key Content	Pacing
1.3 Drawing and Describing Shapes	The main idea is understanding different ways to describe shapes, including names, features, and "recipes" for creating them.	• names, features, and recipes for describing shapes	1–2 days
1.4 Drawing from a Recipe	Students describe drawings in terms of "recipes" for drawing them and interpret recipes to create the drawings. They begin to use precise language to communicate clearly and learn some possibly new geometric vocabulary.	• recipes for describing shapes • geometric language	2 days
1.5 Constructing from Features: Problem Solving	Students use hand construction tools to construct various shapes by their described features. In the process, students learn the difference between a construction and a drawing, learn about the tools, read about the historical importance of geometric constructions, and explore properties of triangles.	• construction vs. drawing • hand construction tools • properties of triangles	4 days
1.6 Constructing from Features: Paper Folding	Students explore paper folding as a construction technique and get a short introduction to origami.	• paper folding as a construction technique • origami as a field of mathematical exploration	2 days
1.7 Constructing from Features: Group Thinking	Students work in groups to create a construction. Each student in a group is given a clue to the object they need to construct. They cannot show the clue to anyone else. Instead, they can read it, restate it, draw it, etc. The focus is on clear communication and on cooperation among group members.	• working as a group to create a construction	3 days
1.8 Algorithmic Thinking: Directions for People	This investigation prepares students for working with Logo's precise commands by asking them to give formal directions for people to follow. Students write directions for moving along paths (following a map from one place to another).	• following directions to trace out a path • writing directions to trace out a path	2 days

Investigation	Description	Key Content	Pacing
1.9 Algorithmic Thinking: Directions for Robots	This is the first investigation using Logo; basic commands are introduced in the context of maps and writing directions. This continuation of the previous investigation moves towards the formal language of Logo.	• algorithmic thinking • describing shapes with recipes • basic Logo commands	3 days
1.10 Algorithmic Thinking: Angles Around a Center	Students are introduced to more advanced programming ideas that allow them to structure their algorithms. They learn that the total measure of angles around a center point is 360°. They use this result in discovering the Total Turtle Turning Theorem, which describes the total angle a turtle must turn in order to start a trip and end a trip in exactly the same position.	• algorithmic thinking • angles around a center point • Total Turtle Turning Theorem	4 days
1.11 Algorithmic Thinking: Spines, Stars, and Polygons	Students are introduced to some of the power of computer programming: generalizing, making one process that covers a variety of situations, and keeping an algorithm fixed but scaling the size of figures.	• algorithmic thinking • exterior and interior angles in polygons • regular polygons	3 days
1.12 Algorithmic Thinking: Irregular Figures	Students must pull together what they've learned about Logo to create irregular shapes.	• algorithmic thinking • using Logo to create irregular figures	2 days
1.13 Constructing from Features: Moving Pictures	Students learn the basics of geometry software while exploring the features necessary to determine some familiar shapes.	• geometry software • constructions	4–6 days
1.14 Warm-Ups	Students are introduced to the idea of looking for invariants, first in the familiar context of tables of numbers and then in a few geometric stiuations.	• concept of an invariant • circumference/ diameter • parallel lines and angles	1–2 days
1.15 Numerical Invariants	Students use geometry software to hunt for numerical invariants: constant sum, product, ratio, and difference in measurements.	• constant sum and product • constant difference and ratio • some geometric invariants	4 days

Investigation	Description	Key Content	Pacing
1.16 Spatial Invariants	Students use geometry software and manipulatives to explore spatial invariants: shape, collinearity, and concurrence.	• spatial invariants (shape, concurrence, collinearity)	5–6 days
1.17 Parallel Lines	Students use geometry software to investigate parallel lines and angle relationships: congruent corresponding, congruent alternate interior angles, and supplementary same-side interior angles. Students use these angle relationships to write a reasoned argument that the sum of the measures of a triangle's angles is 180°.	• parallel lines • angle relationships • sum of angles in a triangle	3 days
1.18 Investigations of Geometric Invariants	Five investigations are provided. In most classes, each student will do two or three of them.		
Midlines and Marion Walter's Theorem	Students use geometry software to explore midlines of triangles and then "Marion Walter's Theorem" (a theorem about connecting trisection points to opposite vertices in a triangle).	• midlines in triangles • investigating a shape and a numerical invariant	3–4 days
A Folding Investigation	Students explore a shape invariant through paper folding: Pick a point on a square and construct the perpendicular bisectors of the segments between that point and each of the square's vertices. What shape will contain the point?	• investigating a shape invariant • perpendicular bisector • extreme cases	2 days
Circle Intersections	Students use geometry software to construct circles and ellipses as the "trace" (locus) of points. Students analyze the constructions to determine why these shapes emerge.	• investigating a shape invariant • segments, circles, and ellipses as invariants	2 days
Centers of Squares	Students investigate a surprising shape invariant: Start with any quadrilateral and build squares facing outward, on each of its sides. If you connect the opposite centers of the squares, the segments will always be perpendicular and congruent.	• investigating a shape invariant • constructing squares • properties of squares	2 days

Investigation		Description	Key Content	Pacing
(1.18)	**Constructing Invariants**	Rather than looking for invariants in given situations, students use geometry software to create situations that contain given invariants.	• independence of area and perimeter • constructing invariants • constant area • constant perimeter • constant ratios	4 days
1.19	**Guess-AND-CHECK**	For some of the problems in this investigation, the guess-AND-CHECK strategy gets you started on the problem, and refining guesses leads you eventually to the right answer. For others, this strategy gives some real insight into the problem, which must then be analyzed some other way.	• guess-AND-CHECK to gain insight into problems	3 days
1.20	**Reasoning by Continuity**	Students examine the difference between continuous and discrete change and then learn to use some big ideas of continuity in solving problems.	• continuous change • Mean Value Theorem (informal)	4 days
1.21	**Definitions and Systems**	This is a fun, hands-on investigation based on a fanciful story. The main idea is that there are geometric systems other than the familiar Euclidean geometry usually studied in high school, and that properties that hold in one system may not hold in another.	• non-Euclidean geometries (spherical, finite)	1–2 days
1.22	**Non-Euclidean Geometries**	The investigation gives students a sense of some mathematical history, both in the building up of theorems from postulates and in the history of non-Euclidean geometry.	• history of geometries	1 day
1.23	**Visualization Exercises**	This activity contains three "warm-up" visualization problems designed to get students thinking about the invariant they will investigate in this section of the module.	• visualization problems	1 day

Investigation	Description	Key Content	Pacing
1.24 Midpoints in Quadrilaterals	Students experiment with connecting midpoints in quadrilaterals in various ways and look for invariants in these situations. This experiment has two purposes: to get students thinking about invariants within a figure and between two figures, and to begin the central investigation of this section of the module.	• investigating invariants within a figure and between two figures	2 days
1.25 What Do You Find Convincing?	Students read and critique two arguments that claim that connecting the midpoints of any quadrilateral, in order, creates a parallelogram. Both arguments are data-driven, but one is clearly more convincing than the other.	• reading and critiquing student arguments	1–2 days
1.26 Finding Other Invariants	Several problems are given as scaffolding for the Midline Theorem which students will need in order to progress through the rest of the module. If students have conjectured the Midline Theorem already, this investigation can be skipped or shortened.	• Midline Theorem • quadrilateral as two triangles sharing a side (the diagonal)	1–2 days
1.27 Making the Right Connections	Students read and critique another presentation about connecting midlines in quadrilaterals. This time, rather than using data as evidence and using the Midline Theorem as a lemma, students try to explain why you would always get a parallelogram.	• explaining and proving vs. data • assuming a result (lemma) to prove something else	1–2 days
1.28 Can You Say More?	This short investigation is designed to start answering the question, "Why prove things?" Students see that proof can convince you that a result will *always* hold, can give you insight into why it holds, and can help you ask and answer further interesting questions.	• Why prove things?	partial day
1.29 The Midline Theorem	Students examine the Midline Theorem more closely. Reasoning by continuity is used to explain (but not prove) it.	• Midline Theorem • reasoning by continuity	1–2 days

ALTERNATE TIMELINES

We offer here three alternative paths through the module, each with a specific emphasis, and all considerably shorter than the main timeline plan. If the full 17 weeks is too long for your class, we hope one of these will suit you. These three alternatives were chosen to meet the most frequent requests of field-test teachers.

No Technology
- **1.1–1.7 (5 weeks)**
- **1.19–1.22 (12 days)**
- **1.23–1.29 (5 days)**

No Technology (9 weeks)

This module and others in the *Connected Geometry* program were designed to make serious use of technology, so both geometry software and Logo are essential to many of the investigations. However, the materials were field tested in schools where computers were not available, and teachers made adaptations. Most followed the schedule suggested here, using "Without Technology" adaptations to the investigations that suggest technology (see the Investigation Notes).

You will need to supplement your class with lessons to cover some of the content missed by skipping the technology-dependent activities: 360° around a center point, internal and external angle sums in polygons, regular polygons, parallel lines and related angles, and looking for invariants. (You should look at the investigations you will skip and decide which contain content needed for future work in your class.)

Short Complete
- **1.1–1.5 (4 weeks)**
- **1.8–1.11 (12 days)**
- **1.13 (5 days)**
- **1.14–1.17 (13 days)**
- **1.18 (choose 1 investigation) (2 days)**
- **1.19–1.22 (9 days)**
- **1.23–1.29 (5 days)**

Short Complete (13 weeks)

Here is a shorter timeline for covering all of the material. It includes significant work from all sections, but skips many extensions and historical connections. It is possible to make this timeline even shorter, depending on how much of each investigation you cover, how quickly your students progress through the beginning problems and learn the technology tools, and how deeply you explore the investigations.

Focus on Invariants
- **1.1–1.5 (4 weeks)**
- **1.14–1.18 (5 weeks)**
- **1.16 (2 days)**
- **1.23–1.29 (1 week)**

Focus on Invariants (10 weeks)

Many teachers want their classes to focus on investigations, exploring mathematical situations to see what they can discover. This timeline allows for exploring all of the geometry software and other investigations, taking them through many extensions. It skips other areas such as algorithmic thinking and most of the non-Euclidean geometry. By the end of these ten weeks, students should be experts at analyzing a geometric situation, looking for invariants, and trying to explain the invariants based on the construction or situation that gave rise to them.

ASSESSMENT PLANNING

What to Assess

- The student can name, describe, and construct several standard geometric shapes, including regular polygons.
- The student can follow directions to create shapes by hand and with geometry software. The student can create directions (in English or in Logo) based on constructions.
- The student can state a conjecture about angle sums in polygons based on the number of sides. The student knows the Total Turtle Turning Theorem and can relate it to external angles and to constructing shapes with Logo.
- The student can state the Triangle Inequality. Given three sidelengths or three angle measures, the student can decide if the given measures determine a triangle.
- The student knows what an invariant is and can describe several examples of geometric invariants.
- In looking for invariants, the student looks at sum, product, difference, and ratio appropriately, as well as looking for shape invariants, collinearity, and concurrence.
- The student can argue that the angle sum in a triangle is constant and the the angle sum in polygons is dependent only on the number of sides.
- The student knows about, and can argue informally for, concurrence of perpendicular bisectors, angle bisectors, and medians in triangles.
- The student can use the guess-AND-CHECK strategy to solve problems with a finite number of possible solutions and to gain insights into other problems.
- The student has a beginning understanding of continuous change and the Mean Value Theorem.
- The student can describe some of the assumptions (axioms) on which Euclidean geometry is based and how different assumptions lead to different geometries.
- The student can articulate some of the needs for proof in mathematics and can describe how proof is different from data support.
- The student can state the Midline Theorem (as a conjecture) and use it to argue that connecting the midpoints of a quadrilateral in order produces a parallelogram.

Notebooks

Throughout the module, students should be encouraged to keep a notebook that contains:

- daily homework and other written assignments
- a list of vocabulary, definitions, and theorems that emerge during classwork and homework

You may want to collect these notebooks periodically for assessment.

QUIZZES AND JOURNAL ENTRIES

Investigation	Journal Suggestion or Presentation	Quiz Suggestion
1.1	*Write and Reflect* Problem 10: Explain the Triangle Inequality and extend it to noninteger lengths.	
1.5	*Write and Reflect* Problem 39: Investigate attempts to trisect angles.	
1.7		Students construct objects from the table given in the Teaching Notes.
1.10	*Write and Reflect* Problem 13: Relate the Logo commands `fd` and `bk` to positive and negative numbers. *Write and Reflect* Problem 18: Explain the Total Turtle Turning Theorem.	
1.11	*Write and Reflect* Problem 12: Write about exterior and interior angles in polygons.	
1.12		Students create the irregular figures with Logo and explain how to do it.

Investigation	Journal Suggestion or Presentation	Quiz Suggestion
1.13	*Write and Reflect* Problem 19: Write directions for creating UnMessUpable figures with geometry software.	
1.15	*Write and Reflect* Problem 12: Argue for constant angle sum in polygons based on an assumed result about triangles.	
1.16	*Write and Reflect* Problem 15: Summarize results about concurrence in polygons. *Write and Reflect* Problem 23: Explain concurrence of perpendicular bisectors in triangles. Problem 24: Explain concurrence of angle bisectors in triangles.	Students summarize different kinds of invariants to look for and clues about when to look for sums, products, differences, or ratios.
1.17	*Write and Reflect* Problem 11: Argue for 180° angle sum in triangles.	
1.19	*Write and Reflect* Problem 2: Explain the best strategy for the “Find Your Homework” game.	
1.20	*Write and Reflect* Problems 4 and 5: Describe continuous vs. discrete change.	Problem provided in the Teaching Notes
1.21	*Write and Reflect* Problem 15: Explore angles in spherical triangles.	Problems 11 and 19
1.22	*Write and Reflect* Problems 1 and 2: Reason about Euclidean and spherical geometry.	
1.26		Problem 2

Investigation	Journal Suggestion or Presentation	Quiz Suggestion
1.27	*Write and Reflect* Problem 1: Critique the dialog explaining the result about quadrilaterals. Take one character's point of view and defend it. Besides the final presentation or write-up of the investigation, you may also want students to launch their own experiment of another way to connect midpoints in quadrilaterals and present their investigation to the class. Both options are explained further in the Teaching Notes.	

Teaching Notes

Investigation 1.1

Student Pages 1–10

PAGE 1

Problem Solving in Geometry

Materials: Each problem has different needs in terms of materials. These include:

- paper and scissors
- clay (or other 3D modeling material)
- knives or dental floss for slicing clay
- Cuisenaire® or other rods
- dice
- protractors and compasses

The day before: Decide which problems your class will tackle first, and have the materials ready.

Overview

This investigation will get students right into solving geometry problems. In the process, they will learn

- some properties of polygons, angles, and circles;
- a lot of geometry vocabulary;
- about nets and cross sections of three-dimensional shapes;
- the Triangle Inequality.

There are no prerequisites to this investigation. Students will construct various shapes using protractor, compass, or drawing templates, but they do not have to know how to do this before working on the problems—figuring it out is part of the problem. For some problems, they will need to know that a diameter goes through the center of a circle and what defines a rectangle.

Teaching the Investigation

Some of these problems work best with an introductory whole-class discussion or activity (see "Notes" below). Many of them are ideal for small-group work. The problems occur in related groups, but there is no need to do the groups in the order presented. After each group of related problems, you may want to conduct a whole-class discussion. Here is one suggested sequence:

Day	Discussion	Homework Suggestions
Day 1	Read Introduction to this section aloud and use this as the basis for an introductory class discussion. Work in groups on the "For Discussion" activity about cylinders.	Problems 1 and 2
Day 2	Discuss Problems 1–2. Work on "The Handshake Problem" and "Diagonals in Regular Polygons."	Write up Problem 4, and write about how Problems 3 and 4 are related.
Day 3	Discuss "Nets" Problems 5–6.	Problem 7

If students are not comfortable reading so much on their own, they might read aloud in class, perhaps stopping between paragraphs to ask about new words and ideas.

Day 4	Work on "The Triangle Inequality" in groups.	Problem 10
Day 5	Make a whole-class formal statement of the Triangle Inequality. Work on "Cross Sections."	
Day 6	Work on "Angles Inscribed in Semicircles," including a whole-class discussion about Problem 15.	Read "And What is Geometry?" and the Perspective.
Day 7	Class discussion: What have we learned? Students work on "Checkpoint" problems.	

If no one in class knows the shape of either item, you might ask students to look at them and then write answers to the two problems for homework.

Problems 1–2 introduce the idea that the geometric properties of a shape can serve practical purposes. Reasonable conjectures, not previous knowledge, are what to look for. For example, a student might theorize that the reason manhole covers are circular is so that they can be screwed into the street like the cap on a bottle. (This theory happens to be false but is consistent with the shape.)

For cacti, storage and conservation are more important than absorption and transpiration, so they are cylindrical or globe-shaped rather than feathery and branching like wet-area plants.

The purpose of the cylinder discussion is to stretch students' mental *pictures* to fit their definitions and to focus attention on the irrelevance of the *measures* of the two essential features (circular base and projection into the third dimension). Cylindrical shapes are common in nature. A nice question for discussion is: Why is this shape so common, and what is so useful about it? The circular cross section is strong and cost/energy efficient. (It helps to reduce the surface area, within other constraints.) In nature, controlling the surface area helps to control loss (or gain) of heat and water.

Problem 3: Strategy 1, trying to figure out how the answer *changes* if one more person entered the room, is the essence of a powerful technique known as mathematical induction, which is useful in many counting problems.

Students who know how the All-Star lineups are done in baseball might explain how each team member runs out and shakes hands with all the others who are already in the line.

One teacher made a nice classroom activity based on Strategy 2. Everybody got up, shook hands with everyone else, and counted the number of times that they, personally, shook hands. Her students thought of multipling the number of handshakes they counted ($n - 1$) by the number of people who were counting (n). After testing their solution on a class with only two people, they knew what to do. The result: $\frac{n(n-1)}{2}$.

Strategy 3 involves drawing a picture. If there are n people in the room, then the handshakes can be modeled by a regular n-gon with all the diagonals drawn in (every person corresponds to a vertex, and every vertex is connected to every other vertex exactly once). This makes a nice connection to Problem 4.

Regarding the side note to Problem 4: "Trigon" and "tetragon" are perfectly valid and are used some places. The difference is Latin versus Greek terms, not right versus wrong terms.

You may need to set some rules as to how the rods meet at the corners.

Problem 8: A powerful strategy for figuring out what triples will make a triangle is to keep one side constant and check what will fit it. Some students may have difficulty organizing their work. Entering their results in a table can help. The blackline master at the end of this investigation's Teaching Notes can be copied to provide structure for your students if you think they would benefit from it.

Problems 12–13: At this stage, students are not likely to be able to give a complete explanation about why these figures must be rectangles. Still, partial explanations should be encouraged and can be collected. The problems can be revisited. A good time is just after students learn that the sum of interior angles in a quadrilateral is 360°. Students may not yet appreciate the importance of proof in mathematics. For many, it may still be too early to raise this issue, but for some this may be a good opportunity to begin discussing the difference between a well-founded conjecture and an established theorem.

What's coming up? Proving occurs throughout this module. The last section of this module, "Beyond Belief," focuses on it explicitly.

See "Resources" on the next page for a gelatin recipe.

Problems 16–20: Various materials will work. A very thick mixture of gelatin was used quite successfully in some of our test classes and is especially good for "seeing inside" the solid before making the cut. One college class used play dough (preschool or kindergarten teachers can supply easy recipes). Potatoes, sponges, and clay slabs also work well. Most of these materials can be turned into ink stamps at the end. Students must first cut their material into cubes. (They need not be perfect cubes, but they should be close.) Students will need several cubes to work with. Dental floss works better than knives for making planar slices through clay.

Problem 23: It is unusual to find ellipses brought up so early in mathematics classes. We introduce this shape, show some places where it is found, and give it a name early, so that as it makes appearances later on it will be recognized as a familiar figure, and its properties can become part of the study. While recognizing the reappearance of the ellipse and knowing its name is important, naming the other conic sections is optional at this point; sketching them is enough.

ASSESSMENT AND HOMEWORK IDEAS.......

The last problem in each related group tends to make a good homework problem. These are also good individual assessments if students solve the initial problems in groups.

ADDITIONAL RESOURCES ....

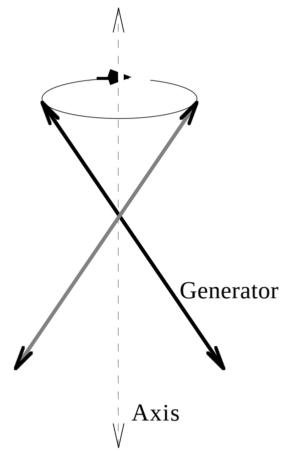
SAFE-T Products (P.0. Box 692, LaGrange, IL 60525) sells scissors that will cut paper but not skin or hair, compasses without the sharp points, and protractors and rulers that won't break and don't have sharp corners.

Here is a recipe for gelatin from Barney Martinez in Daly City, California. He got it from Bev Bos, a preschool teacher who gives workshops around the country.

1. Mix 32 packets of Knox gelatin with 22 cups of hot water.
2. Use containers of various shapes—cubes, cones, cylinders, or paper cups if that's all you can find—and spray the inside with a vegetable oil spray, like PAM.
3. Fill the containers with gelatin, chill, and remove the hardened gelatin from the molds. They will last 2–3 days without refrigeration.

MATHEMATICS CONNECTIONS . . . .

Problem 4: Think of the cone as the surface generated by a line (a generator) that rotates about another intersecting line (the axis of the cone).



A plane perpendicular to the cone's axis cuts circular cross sections. Tilting the plane from perpendicular elongates the cross section to an ellipse. In these positions, the generator of the cone is never parallel to the intersecting plane: picture it rotating around the cone's axis, and constantly cutting the plane. As the plane tilts further, the ellipse lengthens—it takes longer and longer for the intersecting plane to "get back out" of the cone after it has entered. The generator is still always intersecting with the plane, but as it becomes "close to parallel," its intersection with the plane gets quite far away, lengthening the cross section.

If the plane is tilted exactly to the point at which it makes the same angle with the cone's axis as the cone's generator does, then there is exactly one position of the generator that is parallel to the plane. What used to be the "other end of the ellipse" is pushed out to infinity—the plane never gets back out of the cone—and the cross section (formerly elliptical and still approximately elliptical at the "near end") is therefore unending. This cross section is called a *parabola*.

Note the *para* in both *parabola* and *parallel*.

If the plane tilts still further, then there is more than one position (how many?) in which the generator is parallel to the plane as it rotates around the cone's axis. The plane cuts through both branches of the cone, and the cross section therefore has two separate parts: hyperbolic cross sections.

The Connected Geometry module *A Matter of Scale* has a section on creating other self-similar objects with Logo.

Fractal Trees Students may be interested by the "fractal" trees pictured in the Student Module. They are easily created in Logo, and can lead to mathematical ideas such as recursion and similarity. If your students don't already know Logo, you might wait and come back to these procedures later in the module, after the "Algorithmic Thinking" investigations. The set of procedures below leads up to the idea of recursive calls to a function.

First, create a program that will draw branches of various sizes:

```
to Branch :length
 fd :length  bk :length
end
```

Then, express the pictorial plan of the tree (where the branches come off the trunk and at what angles) as a procedure, with an input that allows you to draw the plan at various sizes.

```
to VTree4  :trunksize
 fd 0.35  * :trunksize
  lt 18   branch 0.5 * :trunksize   rt 18
 fd 0.15  * :trunksize
  rt 16   branch 0.4 * :trunksize   lt 16
 fd 0.3   * :trunksize
  lt 20   branch 0.35 * :trunksize   rt 20
 fd 0.12   * :trunksize
  rt 19   branch 0.3 * :trunksize   lt 19
 fd 0.08   * :trunksize
  lt 0   branch 0.5 * :trunksize   rt 0
 bk :trunksize
end
```

You may simply give students these procedures and ask them to try each of them with various inputs. For example, they might try `Branch 10`, `Branch 50`, `VTree4 100`, and `VTree4 50`. Ask students to describe the results, using the ideas of similarity and scale.

To let this tree "grow," first create `VTree3`, absolutely identical to `VTree4` (no change except the title). Then edit `VTree4` and replace `Branch` with `VTree3` everywhere. Ask students to try out `VTree4`. What has changed?

If you create `VTree2`, absolutely identical to `VTree3` (no change except the title),

you can then edit VTree3 and replace all of its branches with VTree2s. Try this. Continue the process another step.

This entire nested structure—things within things within things, all essentially identical except for scale—is a very important mathematical idea known as "recursive structure." It is found in the self-similar designs of nature (for example, the tree here, snowflakes, clouds, and shorelines).

It is possible to capture this entire structure in a single procedure. The result is more compact, but, like terse writing, can be harder to understand until one is quite used to it. Here is what the recursively-defined, single-procedure form might look like. It is given only as an example, and is not sensible to give to students unless they have already developed considerable comfort with Logo (or some other computer language that supports recursive definition) and will continue to develop their programming skills.

```
to VTree  :growth :size
 if :growth = 0 [stop]
 fd 0.35  * :size
   lt 18   vtree  :growth-1 0.5 * :size    rt 18
 fd 0.15  * :size
   rt 16   vtree  :growth-1 0.4 * :size    lt 16
 fd 0.3   * :size
   lt 20   vtree  :growth-1 0.35 * :size   rt 20
 fd 0.12   * :size
   rt 19   vtree  :growth-1 0.3 * :size    lt 19
 fd 0.08   * :size
   lt 0    vtree  :growth-1 0.5 * :size     rt 0
 bk :size
end
```

BLACKLINE MASTER........................

Students should use the Blackline Master that follows to record their results in Problem 8. Each student will need at least one copy of this page.

PROBLEM SOLVING IN GEOMETRY (continued)

Roll			Triangle?
1	1	1	yes
1	1	2	no

PICTURING AND DRAWING

Materials:
- rulers and paper
- photos of famous people

If you have a particular interest in art, this investigation will allow you to do some special work with your students. In addition to the ideas here, you may want to talk about perspective drawing, generating fractals from mathematics, or any other art-geometry connection.

Note that this sequence does not include every problem. It is just one possible way to move through the investigation.

OVERVIEW

This investigation is designed to help students create mental images well enough to analyze their parts and analyze visual scenes well enough to draw them. The specific drawings we ask students to create—3D letters, Escher-like impossible figures, and faces—are designed to appeal to a broad variety of students. Skills include noticing proportions, noticing and representing parallels, and distinguishing between what is seen and what is imagined.

TEACHING THE INVESTIGATION

Here is one suggested sequence:

Day	Discussion	Homework Suggestions
Day 1	Working in small groups, answer the questions in Problems 1 and 2. Follow with a whole-class discussion.	Write up answers to those two problems.
Day 2	Work on Problems 5 and 6. Encourage discussion and modeling.	Problems 7 and 8
Day 3	Students work with partners on Problem 9, drawing each other's faces. Then they work individually on Problem 10. (You may want to have some photos of famous people around the room for students to choose from.)	Problem 11; write about why it's easier to draw the figure partially covered than to draw the whole thing.
Day 4	Work on Problems 14–17.	"Drawing 3D"
Day 5	"Using Pictures to Explain Ideas": Work in small groups on Problems 18–20. Each group will be asked to explain its ideas about one of the problems to the class.	Problem 21
Day 6	Problems 27 and 28	

You can often "watch" students make their mental pictures. Eyes tend to go up and off to the left or right as people create and manipulate mental images.

Problem 2: Students often have difficulty picturing more than one circle through two points (usually they are pictured as the endpoints of a diameter), and they sometimes need prompting to think about the circle and square rotating in space.

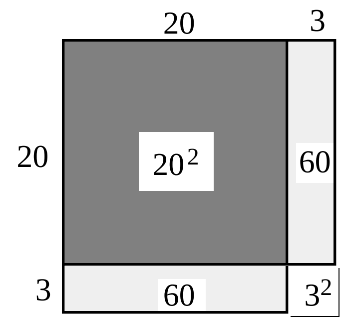
Problem 5: Some students will cut squares out of paper and try to make the shadows. Encourage them to explain how they have to move the square relative to the light in order to change sidelengths and angles.

Problem 11: Drawing from a partially-covered picture helps the student separate out the geometric features from the overall gestalt of the figure. Drawing from memory exercises the student's ability to hold these features in mind and to recall how they are assembled into a whole. In reproducing an "impossible picture" and describing what is wrong with it, students attend to various attributes of the representation of two and three dimensions, such as parallel lines, parallel planes, and how things fit together.

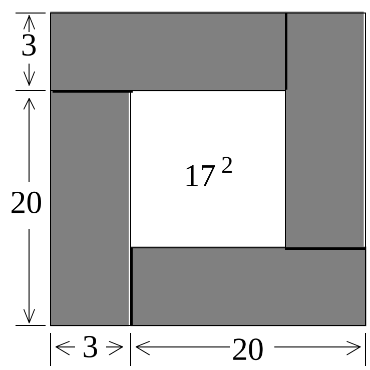
Problems 18–19: Students who have had experience in algebra may find these visual "proofs" quite enlightening. For students who have not had enough experience in algebra to understand the pictures' captions, you may adapt the problems to use specific numbers.

$$(6 + \tfrac{1}{2})(4 + \tfrac{2}{3}) = 6 \cdot 4 + 6 \cdot \tfrac{2}{3} + \tfrac{1}{2} \cdot 4 + \tfrac{1}{2} \cdot \tfrac{2}{3}$$

PICTURING AND DRAWING
(continued)



$(20+3)^2 = 20^2 + 2 \cdot 20 \cdot 3 + 3^2$



$23^2 - 4 \cdot 20 \cdot 3 = 17^2$

Some students, including some who have apparently done quite well with graphing, seem to have no idea where to start in interpreting this picture—perhaps it is too unlike graphs they've worked with.

Problem 20: Encourage students to tell a plausible story connected with the graph. There is no question at all about the relation between terror and time—only a question about where *other* events (like the exam, itself) fall along the time axis.

One group argued about whether circles could be faces and about the "edges" of cylinders.

Problem 27: Asking students to identify vertices, faces, and edges on impossible figures may seem strange, but we have found that when students do this problem in small groups, they generate very interesting discussions. Their understandings and misunderstandings come to light.

Homework assignments are suggested in "Teaching the Investigation."

Assessment and Homework Ideas.......

- Most of these problems are not testing what students have *learned* (they haven't learned how problems like these are to be thought about) and cannot therefore be used for assessment or grading.
- Assessing Problem 21: Are the axes labeled, and do the labels make sense? Are the conventions (independent variable on the horizontal axis, dependent variable on the vertical) observed? Does the graphed relationship make sense? Does the student's explanation of the graph account for it well?

Supplementary Activity ....

Here's a very nice activity from Ros Welchman of Brooklyn College: For each group of four students in your class, prepare a set of small cards. ($3'' \times 5''$ cards, cut in quarters to make $1.5'' \times 2.5''$ cards do very well.) Each set should contain all of the terms on the blackline master, one per card.

Give each group a set of terms and ask students to sort them in some meaningful way. You can leave the task otherwise open or can restrict students to at most six categories. Have students explain their categories and list the terms they included in each. Did any terms seem not to fit *any* category?

There are many reasonable ways to sort the terms. Students might, for example, classify *circle* and *cube* along with other objects as Shapes. Alternatively, two-dimensional objects like *circle*, *net*, and *cross section* might be classed separately from *cube*. Or a curvilinear might be distinguished from straight-line figures. Students often discuss which terms describe *objects* and which describe *qualities* or *relationships*. Among the objects, students also sometimes discuss whether "elementary parts" (*angle*, *point*, and so on) are to be kept separate from objects built from them.

ADDITIONAL RESOURCES . . . .

The optional project suggested in Problem 35 may appeal to students who have a special interest in art. Here are some good references on M.C. Escher:

Bool, F. H. *M. C. Escher—His Life and Complete Graphic Work*. Harry Abrams (publisher), 1992.

Ernst, Bruno. *Magic Mirror of M. C. Escher*. Random House, 1995.

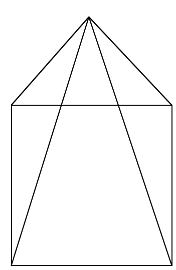
Escher, M. C. and Vermeulen, J. W. *Escher on Escher*. Harry Abrams (publisher), 1989.

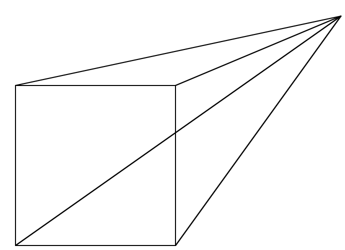
Schattschneider, Doris. W. H. Freeman & Co., 1992.

"The World of Escher" Web site: http://www.texas.net/escher

MATHEMATICS CONNECTIONS . . . .

Problem 5: Here is a very nice model for this problem: Make a square pyramid out of clay. The vertex is a point source of light, the square base is a shadow, and cross sections represent shapes (and their orientation relative to the light) that will cast the shadow. Don't forget a nonright pyramid!





Problem 30: Any new line will cut some of the previously-existing regions into two separate regions each. To maximize the number of regions created by *five* lines, we must start with an arrangement of *four* lines that creates the most regions, and place the fifth line in a way that cuts the largest number of these regions into two parts. The fifth line cannot cut all of the regions. How many can it cut?

When a line is placed down to cut across some regions, each of the regions "covers" a portion of the line: the regions divide the line into lengths, just as the line breaks the regions into subregions. So the question boils down to this: Into how many lengths can this line be divided? It is divided into lengths by intersecting with the other lines—only four of them—that have already been placed. Those four intersection points divide the new line into five lengths, and each of these lengths is the boundary between two new regions that used to be one. Therefore, the *line* divides five previously-existing regions into ten regions. Thus, if four lines divide the plane into 11 regions, then five lines can divide the plane into 16 regions.

Problem 31: If students investigate how the *arrangement* of planes affects the number of regions, they may quickly see that parallel planes divide space into fewer regions than intersecting planes, and that when several planes intersect along the same line, they divide space into fewer regions than planes that intersect in pairs. This is like the result with lines in the previous problem. Students can generally visualize how *three* planes can be arranged to divide space into as few as four and as many as eight regions. However, it is *so* difficult to picture multiple planes in space that few people can visualize the effect of even one more plane. To know what happens with five planes, one must adopt a different strategy, one that is only partially visual.

The strategy developed in Problem 30 can be moved up a dimension. When a new plane cuts n chunks of space into twice that number of regions, it is *itself* cut into n regions, each of which is the dividing wall separating the two new regions of space. Thus, to figure out how many new regions a fourth plane can create, one must look at how the three prior planes can intersect it. Each one intersects the fourth plane in a line, and those three lines can divide the fourth plane into, at most, 7 regions (from Problem 30). So, if three planes divide space into 8 regions, then four planes can divide space into at most seven more, or 15 regions.

PICTURING AND DRAWING
(continued)

This idea can be summarized in a table like the one below.

3 planes can divide space into 8 regions.

3 lines can divide a plane into 7 regions.

So, 4 planes can divide space into 7 + 8 regions.

Number of planes dividing space	1	2	4	8	15	26	42	64
Number of lines dividing a plane	1	2	4	7	11	16	22	29
Number of points dividing a line	1	2	3	4	5	6	7	8
Number of objects of one dimension that divide up an object of one higher dimension	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*

The patterns in the numbers generated by the preceding algorithm and similar ones can be quite interesting and can be explored on spreadsheets or with Logo. Students who are facile in programming in Logo, can model the table with this procedure:

```
to Regions :dimension :dividers
 if :dimension = 0 [op 1]
 if :dividers = 1 [op 1]
 output sum
   (regions :dimensions  :dividers - 1)
   (regions :dimensions - 1   :dividers - 1)
end
```

Rewriting this just in terms of **r**ows and **c**olumns, the entries are:

```
to Entry :r :c
 if :r = 0 [output 1]
 if :c = 1 [output 1]
 output sum (entry :r :c-1) (entry :r-1 :c-1)
end
```

This is quite similar to the algorithm that generates entries in Pascal's triangle.

```
to Pascal :r :c
 if :r = 0 [output 1]
 if :c = 1 [output 1]
 output sum (pascal :r :c-1) (pascal :r-1 :c)
end
```

The powers of two that one sees in the table on the previous page become powers of three when the algorithm is modified slightly.

```
to Pow3 :r :c
 if :r = 0 [output 1]
 if :c = 1 [output 1]
 output sum (pow3 :r :c-1)  2 * (pow3 :r-1 :c-1)
end
```

BLACKLINE MASTER.........................

The vocabulary terms for the "Supplementary Activity" are included on the Blackline Master that follows.

PICTURING AND DRAWING
(continued)

corresponding	heptagon	inscribed	cone
cross section	cylinder	diagonal	cube
equilateral triangle	equilateral	invariant	face
cylindrical	horizontal	segment	base
nonrectangular	octagonal	octagon	line
parallelogram	pentagonal	pentagon	net
quadrilateral	hexagonal	hexagon	radius
rectangular	rectangle	parallel	angle
semicircle	polygonal	polygon	square
triangle inequality	vertical	vertex	point
perpendicular	endpoint	pyramid	circle
trapezoid	triangle	regular	sphere

Drawing and Describing Shapes

Materials: Clay and carving soap are useful for building and testing models in Problems 1–4.

Overview

The main idea of this investigation is to develop clear language for describing shapes. Students learn to use names, features, and "recipes" (algorithms) to describe shapes precisely.

If students have already worked on the problems about shadows in Investigation 1.2, they will be able to get started on Problems 1–4 more easily.

This is a very short investigation, taking only one or two days of class time. Here is one suggested outline for teaching it:

Teaching the Investigation

Before starting the lesson, ask students to read the introduction and do Problem 1 for homework.

Day	Discussion	Homework Suggestions
Day 1	Work on Problems 2–4 in groups, building models of the figures described.	Problems 5 and 6
Day 2	Discuss homework and "recipes," leading into the next investigation.	

What's coming up? The next investigation works more on drawing from a "recipe," and these ideas eventually lead to programming a Logo turtle to draw shapes.

Some ideas for class discussions:

- An overhead projector can be used to demonstrate the shadows cast by various objects in the first four problems.
- Any geometric recipe, just like any cooking recipe, assumes some knowledge on the part of the person following the recipe. For example, calling for a cup of flour assumes the reader knows to use a measuring cup and not a teacup.
- When is it reasonable to interpret such non-specific statements as "turn right" to mean "turn 90° right"? You may want to draw students' attention to the fact

that the way we give and receive directions has a lot to do with our previous experiences and with the context at hand.

That "common shape" is the tip of a screwdriver.

Problem 4: Here is a hint for students: There is a common household tool that has this shape.

ASSESSMENT AND HOMEWORK IDEAS.......

Homework ideas are given above in "Teaching the Investigation." Assessment at the end of the next activity would be appropriate.

ADDITIONAL RESOURCES ....

Shadow Play: Making Pictures with Light and Lenses by Bernie Zubrowski (New York: Morrow, 1995) contains interesting ideas for extensions. The book describes investigations with shadows and light, explains the difference between shadows from the sun (parallel rays) and from most artificial sources (diverging rays or diffuse light), shows how to make a "shadow box" for experiments, and more.

Teaching Notes

Investigation
1.4

Student Pages 27–28

DRAWING FROM A RECIPE

Materials:

- copies of blackline masters
- compasses
- rulers
- protractors

The day before: Copy blackline masters, cut the copies into four parts, one picture per part. Each student gets a single picture to work on.

What's coming up? These investigations will lead nicely into the kinds of descriptions of shapes necessary to write Logo programs.

One teacher decided to review vocabulary from Problem 1 before students worked on it. Using themselves and objects around the room, students were asked to "act out" words like perpendicular, midpoint, tangent, and diameter.

OVERVIEW

The main ideas:

- Describing drawings in terms of "recipes" and interpreting these recipes to create the drawings.
- Using appropriately-precise language to communicate clearly in these recipes.
- Geometric ideas and vocabulary including: parallel and perpendicular lines, tangent, vertical, horizontal, intersection, endpoint, midpoint, vertex, line segment, equilateral triangle
- Labeling points and segments

A lot of vocabulary is used in this investigation, but none of it is prerequisite. It can be developed and explained as necessary as students work on the problems. If they work in groups, students can help each other to interpret any vocabulary they don't know. The final outcome of the drawings will help them check if they interpreted the words correctly.

Hand construction tools (ruler, protractor, compass) are introduced more formally in the next investigation. For the current problems, let students create the drawings in any way they can, using whatever tools and methods are most familiar. Any difficulties they encounter will motivate learning more sophisticated construction methods later.

TEACHING THE INVESTIGATION

Many of these problems would work well either in class or as homework. Here is one possible sequence through the investigation:

Day	Discussion	Homework Suggestions
Day 1	Discuss "recipes" from the previous investigation; Problems 1 and 2	Problems 3–6
Day 2	Students exchange directions from Problem 6 with a partner to test them out. They get feedback from their partner and revise their directions accordingly. Work on Problem 7 and exchange directions with a partner again.	Finish the work on Problem 7, including revising the directions based on a partner's feedback.

ASSESSMENT AND HOMEWORK IDEAS.......

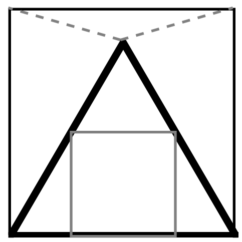
If you use portfolio assessment, both copies of directions could be included, along with a short note from the student about what changes were made and why.

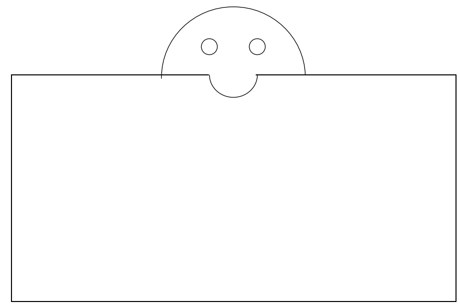
- Problem 7 is an ideal final assessment, especially if students are given the opportunity to revise their work based on exchanging directions with a partner.
- The end of this investigation might be a good time to check on some vocabulary and properties of shapes. You could ask students to illustrate or explain any of the following: perpendicular, midpoint, tangent, diameter, endpoint, equilateral, radius, arc.

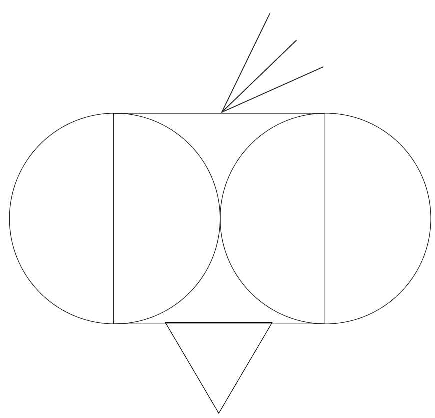
BLACKLINE MASTERS ....

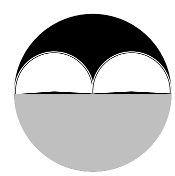
Students will create recipes for the pictures on the following two pages.

DRAWING FROM A RECIPE *(continued)*

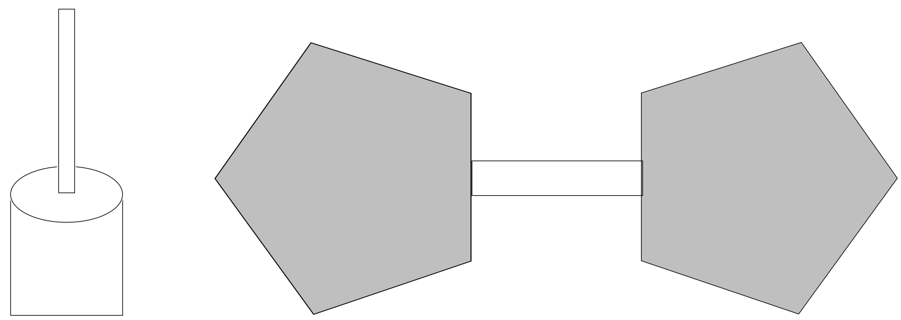


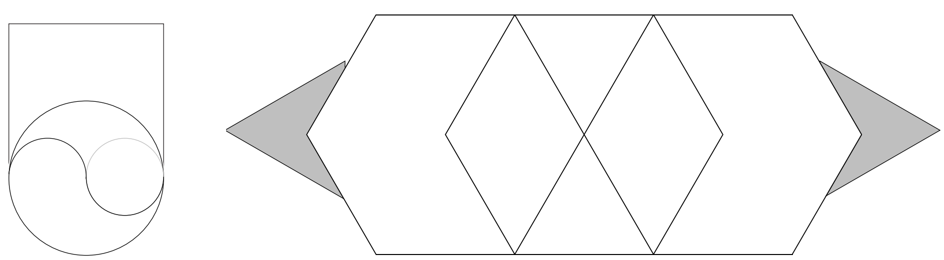






DRAWING FROM A RECIPE *(continued)*





Teaching Notes

Investigation **1.5**

Student Pages 29–43

CONSTRUCTING FROM FEATURES: PROBLEM SOLVING

Materials: A variety of standard and nonstandard drawing and construction tools can be used. These include:
- **compass**
- **straightedges or rulers**
- **protractors**
- **tracing templates (triangles, squares, curves, jar tops, etc.)**
- **sticks of some kind (plastic drinking straws, popsicle sticks)**
- **string**
- **pipe cleaners**
- **stiff paper or cardboard**

What's coming up? Later, students will do constructions using geometry software rather than hand tools, but similarities abound. For example, the circle tool is used to copy lengths.

Suggestion: Have a class discussion about the various tools and what they do, addressing questions like, "How do you 'copy distances' with a compass as it says in the Student Module?"

OVERVIEW

Students construct various shapes by described features, using hand construction tools.

The main ideas:

- Use of hand construction tools
- The difference between a construction and a drawing
- The historical importance of constructions in geometry
- Properties of triangles, including SSS for congruence, AAA for similarity, constant ratios in similar triangles, impossible specifications (violation of triangle inequality, angle sum $\neq 180^\circ$, and so on), and two congruent angles imply two congruent sides

A lot of vocabulary is introduced here in the context of constructions. This helps students to process definitions better than by simply reading them. The vocabulary is *not* prerequisite for the investigation.

TEACHING THE INVESTIGATION

Working through the full investigation will probably take several days of class time. If your students are relatively comfortable with the tools, you may choose to skip some problems and move more quickly. Below is one possible plan.

Before starting the investigation, ask students to do Problem 1 and read about constructions and the hand construction tools as homework.

Day	Discussion	Homework Suggestions
Day 1	Students explain their solutions to Problem 1. Work individually or in groups on Problems 2 and 5. Compare the results as described in Problems 3 and 6.	Problems 4, 7, and 8
Day 2	Discuss results from the previous day (including the "For Discussion" prompts). Work on Problems 9 and 10. Discuss how these problems relate to compasses.	Problems 11–14

If you choose to do all of these problems, it may take more than one day of class and homework. If you choose to skip around a bit, be sure to include the following problems: 16, 18, 20, 21, and 23–25.

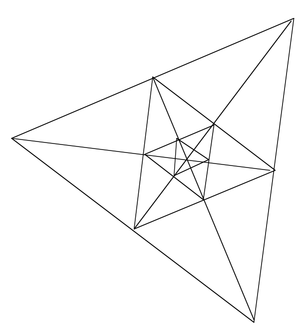
Day 3	Discuss homework. Work on problems from "Other Constructions."	Complete "Other Constructions" and read "A Mathematical Memoir."
Day 4	Try Problem 32. Read "Impossible Constructions."	

Note that the section on "trisecting an angle" is more open, and hence possibly more difficult for some groups to get started on. You may take that into consideration in your assignments.

From here, you can proceed with the entire class working through two or three of the "impossible constructions." Alternatively, you could divide the class into groups, with each group working through one section and presenting its results.

Paper folding is an often-overlooked construction tool. The symmetry of folds provides an ideal way to construct midpoints (match the two endpoints and fold; the crease is at the midpoint) and angle bisectors (match the two rays and fold; the crease is itself the angle bisector). This method also allows you to create 90° angles by bisecting a straight angle.

For Problem 27 ("triangles within triangles"), one student found a clever shortcut: Drawing the three medians of the large triangle automatically finds all of the midpoints. (Once you've drawn a smaller triangle, the intersections of its sides with the medians are the midpoints, so connect them to draw another triangle. The intersections of *its* sides with the medians are the midpoints, so connect them, and so on.)



Homework ideas are given in "Teaching the Investigation."

ASSESSMENT AND HOMEWORK IDEAS.......

- The "Checkpoint" problems would make a good cumulative assessment.
- Another assessment option would be to pull some problems from the "Other Constructions" section and create a short quiz. The problems could be left as they are or altered slightly.
- If you choose to have students present the "impossible constructions" in groups, this would make an ideal assessment. You could require individual write-ups in addition to a group presentation.

Supplementary Angle and Protractor Activities

Students often face problems with the use of protractors. This investigation requires students to measure angles accurately and to make drawings that require protractor use. Here are some supplementary activities that may help your students gain familiarity with angles and skill with the protractor. The activities are not all "mathematical," but the students can have fun with them while learning how to use protractors.

Familiarity with angles The students brainstorm everything they know about angles. They may list words like *obtuse*, *right*, and *degrees* and try to explain their meanings through other words, pictures, or actions. (For example, a person may turn halfway around to show 180°.)

Computing angles from the full circle Students find the angle measurement for each angle on Pattern Blocks (see "Additional Resources") by placing several of the same kind of vertex around a point to complete the 360°and counting how many blocks it takes. Then they divide to find the angle measure for that vertex. (Some of the angle measures are not factors of 360°, so students must use some ingenuity to find those measures.)

Some students may call out things like "2,000°," which leads to a nice discussion of angles as rotations. Students generally figure out how to sweep out a full circle for every multiple of 360°.

Estimating angles In pairs, students are called up to the front of the room. Each student in the pair is given a "ray" (a wooden dowel or yardstick). The class gives the students an angle measurement, and they show that angle with the two "rays." They have to come up with a way to distinguish 360° from 0°.

Measuring angles with a protractor Draw several angles on the board. Give students protractors and ask them to measure the angle without any further directions on using the protractor. Because they already have an idea of how to estimate the measure of an angle, they have a basis for trying to figure out a way of getting that result.

Writing directions for the use of a protractor Alone or working together, students write directions for using a protractor.

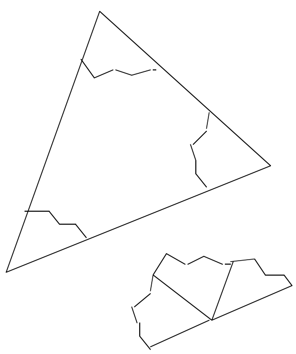
Things to measure include the angle a chair leg makes with the floor, angles on signs up on the walls, etc.

Protractor scavenger hunt In groups of 3 or 4, the students are given a protractor and asked to measure several angles around the classroom and the building.

The chair Place a folding chair on a desk. Ask students to use their protractor to measure every angle they can find on the chair. They must clearly describe which angle it is and write down its measurement.

Shapes See the worksheet in the Blackline Masters at the end of the notes for this investigation.

It is actually important to rip rather than cut the corners off. It allows you to keep track of what was a vertex of the triangle and what was not.



If your students have not had this experience in earlier grades, let them rip off the corners of a triangle and arrange them so that the vertices meet at a point to show their sum—they end up forming what appears to be a straight line. Provided that they recognize a line as a "straight angle" (180°), they may then conjecture that the angle sum of a triangle is 180°, confirming protactor results. Is this a proof for all triangles? No. That will come later. Similarly, students can rip the corners off of any quadrilateral and fit them around a center point to fill 360°. (With concave quadrilaterals, however, students may have difficulty figuring out how to place the vertices.)

ADDITIONAL RESOURCES ....

The Mirror Puzzle Book, Tarquin Publications, England 1985. (Published also in Chinese, Korean, and French) (Honorable Mention, New York Academy of Science Children's Book Award Program)

Pattern Blocks are available from many distributors of mathematics manipulatives, including:

Creative Publications	1-888-MATHFUN
Cuisenaire	1-800-872-1100
Didax	1-800-458-0024

Goniometers (the "gon" is the same root, meaning angle, as in "pentagon" and "hexagon") are another angle measuring device. Built out of two pieces hinged together with something like a protractor at the hinge, goniometers can be used more easily than protractors in many situations. The goniometer can be opened or closed to match some angle to be measured—for example, the angle of bend at your knee—and the angle can then be read directly from the attached protractor.

Many companies sell scissors that will cut paper but not skin or hair, protractors and rulers that won't break and don't have sharp corners, and compasses without the sharp points. SAFE-T Products (P.O. Box 692, LaGrange, IL 60525) is one.

MATHEMATICS CONNECTIONS

The source for this memoir is found in a sidenote for this lesson in the Student Module.

The following paragraph was part of Marion Walter's essay, but we thought it would be more interesting for teachers than for students to read:

> "The academic thing that I remember best about this school was its projects. These involved the whole school for what I think was a week, though it may have been only a few days. Each year a different topic was chosen. I recall two of them: Grains and Birds. Grains was the topic when I was nine. What fascinated me were the pie charts showing different amounts of grain grown locally and in different countries. I don't recall what my contribution was, though I know that *everybody*, whatever age, contributed. "Birds" I recall because I "had" the nightingale and I had to make a small presentation. Can we learn something from the fact that these projects really stand out 50 years later?"

Constructions are often described as if they were just very precise drawings, but precision isn't the issue. The essential element of a construction is that it is a kind of "proof"—a demonstration of how, *in principle*, certain figures can be accurately drawn with the permitted tools. We say "in principle," because mathematics deals with abstract tools, not real ones. A construction executed with real tools (instead of the abstract "tools of the mind") cannot be perfectly accurate: pencil thicknesses, straightedge unevennesses, the inability to copy a distance precisely with a compass, all contribute to the real-life inaccuracies, and make this "actual" construction no more accurate than a meticulously made drawing. And a construction knocked off carelessly on a chalkboard is no less a construction. It shows *how* the drawing can be made precisely, even though it, as a drawing itself, is rough and crude.

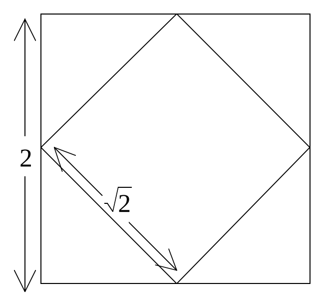
The Poincaré quotation in the sidenote of the Student Module continues, "It is said that geometry is the art of applying good reasoning to bad drawings. This is not a joke but a truth worthy of serious thought. What is meant by a poorly-drawn figure? It is one in which proportions are changed slightly or even markedly, where straight lines are shown as crooked, or where circles acquire incredible humps. But none of this really matters. A poor artist, however, must not represent a closed curve as if it were open, or three concurrent lines as if they intersected in pairs."

CONSTRUCTING FROM FEATURES: PROBLEM SOLVING *(continued)*

Problem 5: The triangle of Problem 5d cannot be drawn on a plane, but *can* be drawn on a sphere! Let vertex A be on the equator. Let vertex B be $\frac{1}{4}$ way round the globe from A and also on the equator. And let the third vertex be at the North Pole. This triangle has three 90° angles!

Problem 18b: The Solution Resource explains only why the result has half the area. If you also care to prove that the result is square, the easiest argument is probably by symmetry: any way the creased figure is rotated, all of its parts (including all the angles) match up, because they were all created in the same way from a set of original parts that are all identical.

This problem implicitly asks students to construct the length $\sqrt{2}$. What about the constructability of irrational lengths? You may start students thinking about that issue now by asking them: "Compared to the original square, what is the length of the sides of the new square?" Define the length of a side of the original square as 2 units; then the legs of the corner triangles (outside the half-sized square) are 1 unit long. The Pythagorean Theorem then says that the length of the hypotenuse is $\sqrt{2}$. This could also be reasoned by area. If the original square's sides are defined as 2, its area is 4 square units. The half-sized square's area is therefore 2 square units, so its side must be $\sqrt{2}$ units.



Now you have a method of constructing one irrational length. Not all irrational lengths *can* be constructed, and the proof of that is also the proof of the impossibility of squaring the circle and many of the other classic impossible constructions.

BLACKLINE MASTERS ....

The worksheet for "Supplementary Angle and Protractor Activities" follows.

Construct:

1. A parallelogram with at least one angle of 50°.

2. A square with at least one side 7 inches long.

3. A rectangle with at least one side 7 inches, but with less area than the square you just drew.

4. An equilateral triangle with an 8-cm side.

Malika says there's no such thing as a triangle with the biggest possible angle. Is she right?

5. A triangle with one angle that's as big as you can make it. How big is that angle?

6. Use just your ruler and pencil to draw a pentagon (it doesn't have to be regular). Measure each angle and write the measurement inside the angle. What is the sum of the angle measurements?

7. Use just your ruler and pencil to draw a hexagon. Measure each angle and write the measurement inside the angle. What is the sum of the angle measurements?

8. Complete this table.

Type of Polygon	Sum of Angle Measures
Triangle	
Quadrilateral	
Pentagon	
Hexagon	

Can you find a rule for the total angle measure of a shape based on the number of sides it has?

9. Draw a 180° angle. What does it look like (describe it in words)?

10. Try to draw an angle that is more than 180° and describe it.

Constructing from Features: Paper Folding

Materials: origami paper or patty paper

Overview

Students explore paper folding as a constructing technique, including a short introduction to origami.

The main ideas:

- Continued emphasis on basic ideas and terms
- Paper folding to create lengths, bisect angles, and so on
- Symmetry
- Introduction to origami as a field of mathematical exploration

Some geometric vocabulary introduced earlier in this module—including *right angles*, *45° angles*, *square*, and *kite*—is used in the directions for folding.

The length $\frac{\sqrt{3}}{2}$ as the sidelength of a square with area $\frac{3}{4}$ is discussed. This may be a difficult idea for students with weaker algebra backgrounds.

This investigation requires a lot of reading and following directions. If your students have difficulty with this, you may want to have them work in small groups on the paper folding constructions so that they can help each other understand the instructions. For such students, you probably do not want to assign one of the constructions for homework as suggested below, because working on it alone may cause frustration.

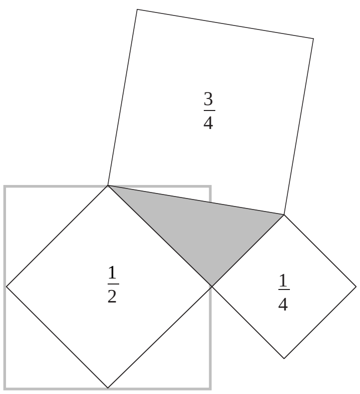
The day before: Be sure to fold the flapping bird at least once on your own so that you are comfortable with the directions and can help students if they have difficulty.

Teaching the Investigation

This is a relatively short investigation. One two-day plan follows.

For homework the night before, ask students to read the introduction and to work on Problems 1 and 2. Also have them read "Origami Mathematicians."

Day	Discussion	Homework Suggestions
Day 1	In class, students follow the directions to fold the flapping bird and answer Problems 3 and 4. Discuss previous work on folding squares.	Work on Problem 5. (This is a hard problem, but students should come up with a solution that seems reasonable to them.)
Day 2	Discuss Problem 5 and the "Ways to Think About It" that follows. Discuss what other lengths might and might not be constructed with folding techniques.	Work on Problem 6 and discuss the "Ways to Think About It" that follows.



$\frac{1}{2}$ size and $\frac{1}{4}$ size squares

Problem 5: Be sure students have already solved Problem 18 from Investigation 1.5 before working on this problem. You may want to ask them as a class to recall their solutions to that problem before beginning this one. Some honors classes that field-tested this *Connected Geometry* module used the following extension problem:

> For what fractions $\frac{m}{n}$ is it possible to fold a square with area $\frac{m}{n}$ of the original? For which fractions is it impossible?

This problem is explored in the "Mathematics Connections" section for this investigation on page 34.

ASSESSMENT AND HOMEWORK IDEAS.......

Again, you should try to fold the objects before giving the assignment to students. Some of them can be very difficult to create!

- Homework ideas are suggested in "Teaching the Investigation."
- Students may try following directions from an origami book to create something in addition to the flapping bird.
- As a project, students might work on the challenge problem discussed in "Mathematics Connections" on page 34. It is a difficult problem but one which many students can start on by trying different fractions to see if any problems arise. Many students will conjecture that it is possible to create squares with area $\frac{m}{n}$ as long as $\frac{m}{n} < 1$, but few will come up with the proof.

ADDITIONAL RESOURCES . . . .

Slightly different instructions for folding the flapping bird appear on pp. 179–182 in Martin Gardner's *The Second Scientific American Book of Mathematical Puzzles and Diversions* (New York Simon and Schuster, 1961).

Patty paper is available from many publishers of mathematics manipulatives, including Key Curriculum Press. You might also ask a local butcher to sell you the paper, which is used to separate hamburger patties. Waxed paper makes nice creases, but it is a bit harder to handle, and you must cut it into squares. Origami papers also work, but are more expensive.

The book *Patty Paper Geometry* by Michael Serra (Key Curriculum Press, Berkeley, 1994) has many additional folding activities that you might want to do with your students. It is targeted to middle and high school geometry students.

Row, T. Sundara. *Geometric Exercises in Paper Folding*. New York: Dover, 1966 (originally published by The Open Court Publishing Company in 1905) includes some classic geometric constructions and theorems explored through paper folding, and it includes more advanced constructions than Serra's.

If you or your students want to do more with origami, here are some good beginning books: *Easy Origami* by John Montroll (Dover; Mineola, NY; 1992). *Favorite Animals in Origami* by John Montroll (Dover, 1996). *The Origami Workshop* by Gay Merill Grass (Friedman/Fairfax Publishers, New York, 1995).

In the United States, the best source of information about origami is the Origami USA society. Aside from selling paper and books by mail, they publish a newsletter with information about local origami groups around the world. They also organize an annual convention in New York City that draws over 500 people from around the world for a fun-filled weekend of folding and sharing. You can write to them at the following address.

Origami USA
15 West 77 Street
New York, NY 10024-5192

You can also find lots of interesting information about origami on the Internet. Here are some of our favorite web sites:

- http://www.terra.net/sasuga/origami1.html
  A source for ordering origami videos and books, as well as hard-to-find origami books and magazines from Japan
- http://www.fascinating-folds.com/paper/
  Another source for origami books as well as a variety of hand-made paper
- http://www.datt.co.jp/Origami/
  A great site for viewing origami photos, finding out the latest origami news, and downloading origami diagrams
- ftp://rugcis.rug.nl/origami/archives/
  An archive of the postings from the origami newsgroup

MATHEMATICS CONNECTIONS . . . .

For what fractions $\frac{m}{n}$ is it possible to fold a square with area $\frac{m}{n}$ of the original? For which fractions is it impossible? It turns out that, for all $\frac{m}{n} < 1$, you can construct a square with area $\frac{m}{n}$ using just folding techniques (assuming you start with a square paper of area 1).

One honors class at Brookline (Massachusetts) High School explored this problem for a week, and came up with the following proof:

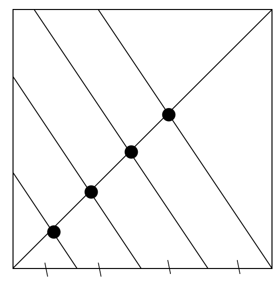
LEMMA 1

You can n-sect the side of your square, so you can create a length of $\frac{m}{n}$ for all $m \leq n$.

You can choose the length of these equal segments, so you can be sure of fitting *n* of them along the diagonal for any *n*.

The class decided that folding the paper over on itself, though it works to trisect segments, was unworkable for general n. They came up with a method to n-sect the side by using parallels. First, construct the diagonal of the square, and mark off n equal lengths along the diagonal (lengths are easily copied with folding). Connect the last endpoint to a vertex of the square not touching the chosen diagonal. Fold parallels

to this segment from each mark along the diagonal. This will n-sect the side of the square.



Most methods for folding parallels require folding a perpendicular (not shown here).

LEMMA 2

You can form a length of $\frac{\sqrt{n}}{n}$, which is the same as $\frac{1}{\sqrt{n}}$.

We know we can do this because of Lemma 1: simply *n*-sect two adjacent sides of the square and join up the first marks of each.

If we create a right triangle with leg lengths of $\frac{1}{n}$ and $\frac{1}{n}$, we can use the Pythagorean Theorem to find the hypotenuse: $(\frac{1}{n})^2 + (\frac{1}{n})^2 = \frac{2}{n^2}$, so the hypotenuse has length $\frac{\sqrt{2}}{n}$.

Now we can create a right triangle with leg lengths of $\frac{1}{n}$ and $\frac{\sqrt{2}}{n}$. Again, we can use the Pythagorean Theorem to find the hypotenuse: $(\frac{1}{n})^2 + (\frac{\sqrt{2}}{n})^2 = \frac{3}{n^2}$, so the hypotenuse has length $\frac{\sqrt{3}}{n}$.

We can continue this process, as long as the lengths fit on our 1×1 square. We see that, in particular, we can form the length $\frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$ by repeating the process $n - 1$ times.

LEMMA 3

You can form a length of $\sqrt{\frac{m}{n}}$.

A similar procedure allows us to form a length of $\sqrt{\frac{m}{n}}$. First, create the length $\sqrt{\frac{1}{n}}$ as described in Lemma 2. Now, create a right triangle with both legs of that length. We

can calculate the length of the hypotenuse: $(\sqrt{\frac{1}{n}})^2 + (\sqrt{\frac{1}{n}})^2 = \frac{2}{n}$, so the hypotenuse has length $\sqrt{\frac{2}{n}}$.

Now create a right triangle with legs of length $\sqrt{\frac{2}{n}}$ and $\sqrt{\frac{1}{n}}$. The hypotenuse will have length $\sqrt{\frac{3}{n}}$. In this way, we can create any length of $\sqrt{\frac{m}{n}}$, as long as it fits on our 1×1 square.

THEOREM

Once you create the length of $\sqrt{\frac{m}{n}}$, simply copy that length to two adjacent sides of your square, and fold a square with that sidelength. Its area will be $\frac{m}{n}$.

Teaching Notes

Investigation **1.7**

Student Pages 59–60

Constructing from Features: Group Thinking

Materials:
- clues and index cards
- drawing tools

In school, in work, in sports, and in one's personal life, it is important to be able to work with others. This activity requires students to work with their group, to communicate, to make sure everyone participates, and to agree upon a solution.

The day before: Duplicate the puzzles. Cut each set into six separate cards.

Before day 2, prepare index cards using the list of terms that appears in the "Blackline Masters," one per index card.

Overview

The format of the following set of ten puzzles is adapted from *Get-It-Together*[1], and is designed to increase the participation of students who tend to leave themselves out or be left out of discussions. Each puzzle consists of a set of six clues that contribute to creating a drawing. The students are to share their clues without actually showing them to each other, and the members of the group are to cooperate to create the drawing.

The main ideas:

- Seeing more than one way to solve a geometric problem
- Working in a group to create a construction
- Assessing students' understanding of the hand construction techniques from the past two investigations

This investigation assumes familiarity with basic geometric vocabulary used in this module (e.g., line segment, angle, parallel, perpendicular).

Teaching the Investigation

Day	Discussion	Homework Suggestions
Day 1	Create groups of four to six students. Pass one puzzle to each group, dividing the cards among the group members. If there are fewer than six students in a group, you can give some students two cards. When a group completes its puzzle, they can work on another one.	Problem 4 (Optional)

[1] ©1989, produced by the Equals project at the Lawrence Hall of Science, Berkeley, CA

Day 2 "Checkpoint" problems: Students who finish quickly can be given another object to work on while the rest of the class finishes. Alter the list of objects to suit the needs of your class. (Polygons, arcs of circles, and midlines of quadrilaterals are all possibilities.) You can also include on each card a list of "illegal words" which cannot be used in the students' step-by-step directions.

Day 3 Groups try out each other's clues from Problem 4. Students may need to revise their work based on what the groups construct.

After the "Checkpoint" problems, there are many avenues for discussion. Some questions you can pose include:

- Which objects from the index cards were easy to draw just by knowing the definitions?
- Which objects were hard to draw, even though you knew what they should look like?
- Were any of the objects impossible to draw with only straightedge and compass?
- If any of the objects was not reproduced correctly by the partners, discuss why. What additional directions would have helped?

Problem 4: It's often easiest to draw the figure first, then write all the directions, and then find a good way to split them up.

ASSESSMENT AND HOMEWORK IDEAS.......

- Because this investigation requires students to talk about what they know, it is a good way to assess students' understanding of the construction techniques from the last two investigations, their strategies for dealing with the unfamiliar, and their ability to integrate their ideas.

- Problem 4 can be done for homework. The opportunity to revise students' directions after some additional work in the activity would make a good final assessment.
- Use some of the objects given in the table on the "Constructions" Blackline Master as in-class construction assignments, some for homework, and some as an in-class assessment.

BLACKLINE MASTERS . . . .

The clues for the 10 puzzles follow (pages 40–44). The constructions for the "Checkpoint" problems are on p. 45.

Puzzle 1: What's the drawing?

Clue 1: This figure uses line segments of three different sizes.

Clue 2: Two of the line segments in this figure bisect each other.

Clue 3: It divides the plane into five regions: the outside, and four inside regions.

Clue 4: This figure uses six line segments. At least two of these are parallel to each other.

Optional Clue 1: The shortest two segments are the same size. The longest two segments are each twice that size.

Optional Clue 2: This figure contains two segments that cross each other, but are not perpendicular.

Optional Clue 3: There is at least one pair of perpendicular line segments.

Puzzle 2: What's the drawing?

Clue 1: This figure uses three line segments. It doesn't cut the plane into separate regions (no "outside" or "inside").

Clue 2: One line segment goes roughly from the northeast to the southwest of the figure.

Clue 3: Two line segments are parallel to each other.

Clue 4: Only one segment is not horizontal.

Clue 5: There is at least one horizontal line segment. If you draw a vertical line through either of its endpoints, the vertical will go through one endpoint of each of the other line segments in this figure.

Clue 6: Each line segment intersects with at least one other. The segments intersect only at endpoints. (That is, no segment crosses another.)

Puzzle 3: What's the drawing?

Clue 1: Only one segment is not horizontal.

Clue 2: Only three segments are horizontal.

Clue 3: One line segment in this figure is perpendicular to all the other line segments in the figure.

Clue 4: Only one line segment has both of its endpoints on another line segment.

Clue 5: This figure contains three parallel line segments.

Clue 6: All segments are the same length. This figure does not divide the plane (no "outside" or "inside").

Puzzle 4: What's the drawing?

Clue 1: One line segment in this figure is four inches long. No other line segment in this figure is more than two inches long.

Clue 2: Two of the line segments in this figure are the same length.

Clue 3: Each segment in this figure meets at least one other segment at each of its endpoints (as in a polygon), but the figure does not cut the plane into separate regions (no "outside" or "inside").

Clue 4: In this figure, there are exactly three places where line segments meet at their endpoints. One of these three vertices bisects the segment connecting the other two.

Clue 5: Two of the line segments of this figure lie directly on the third segment of the figure.

Clue 6: The total length of all of the line segments in this figure is eight inches.

Puzzle 5: What's the drawing?

Clue 1: This figure has four vertices.

Clue 2: This figure has five line segments.

Clue 3: This figure divides the plane into three regions: an outside and two inside regions.

Clue 4: At least one vertex has three line segments meeting at it. All three of those line segments are the same length.

Clue 5: There is at least one pair of parallel line segments.

Clue 6: The two "inside regions" are the same size and shape.

Puzzle 6: What's the drawing?

Clue 1: This figure has six line segments.

Clue 2: This figure contains a square.

Clue 3: At least one vertex in this figure has three line segments attached to it.

Clue 4: This figure contains an equilateral triangle.

Clue 5: This figure divides the plane into three regions, one outside and two inside.

Clue 6: This figure has five vertices.

Puzzle 7: What's the drawing?

Clue 1: This figure is not a polygon and it is not a letter of the alphabet.

Clue 2: This figure has one more line segment than it has vertices.

Clue 3: You can erase three line segments from this figure to leave a polygon.

Clue 4: This figure contains five right angles.

Clue 5: You can erase one line segment from this figure to leave a polygon.

Clue 6: You can erase two line segments from this figure to leave a square.

Puzzle 8: What's the drawing?

Clue 1: All the diagonals of this figure are the same length.

Clue 2: This figure is a polygon and its diagonals.

Clue 3: Half of this figure's line segments cross other line segments.

Clue 4: There are only two different lengths used in creating this figure.

Clue 5: Four segments meet at each vertex.

Clue 6: This figure uses ten line segments.

Puzzle 9: What's the drawing?

Clue 1: All the diagonals of this figure are the same length.

Clue 2: This figure consists of a polygon and its diagonals.

Clue 3: At least two line segments in this figure are parallel.

Clue 4: Three different lengths are used in creating this figure.

Clue 5: If you erase one vertex and all the line segments that connect to it, what remains is a triangle.

Optional Clue: There is a right angle in this figure.

Puzzle 10: What's the drawing?

Clue 1: This figure has six edges (line segments connecting two vertices).

Clue 2: Four of the vertices have only one edge connected with them.

Clue 3: Three edges meet at vertex B. A different three edges meet at vertex C.

Clue 4: This figure has seven vertices.

Clue 5: Exactly two edges meet at vertex A. The angle between them is about 60°, the bisector of that angle is vertical, and the vertex is at the bottom.

Clue 6: If you erase vertex A and the two edges that meet at it, the figure becomes two separated V-shapes.

Constructions

line segment 10 inches long	midpoint of 10-inch line segment
two parallel lines	endpoints of 10-inch line segment
two parallel lines 3 inches apart	two perpendicular lines
acute triangle	obtuse triangle
right triangle	isosceles triangle
equilateral triangle	isosceles right triangle
Trisect a 50° angle. (Divide it into 3 equal-size angles.)	Draw a triangle; then inscribe a circle in it. (Draw the biggest circle that fits inside the triangle.)
Draw a triangle; then circumscribe a circle around it (a circle with all three vertices of the triangle on the circle).	perpendicular bisector of a 10-inch line segment
rectangle	square
ray	circle with 3-inch radius
Divide a 10-inch line segment into 3 equal pieces.	50° angle
acute angle	obtuse angle
Bisect a 50° angle.	trapezoid
tangent to a circle from a point on the circle	tangent to a circle from a point outside the circle
parallelogram	rhombus
any concave quadrilateral	line

Teaching Notes

Investigation **1.8**

Student Pages 61–63

Algorithmic Thinking: Directions for People

Overview

In the next four investigations, students write and analyze recipes that describe shapes (like hexagons) or paths (like a map from one place to another). Students acquire a formal language (Logo) for describing geometric objects, a language that they can apply in later geometry work.

This investigation prepares students for formal language by asking them to give precise directions using their own (natural) language. The main ideas:

- following directions to create a path
- developing and writing directions to trace out a path
- understanding the concept of an algorithm.

Teaching the Investigation

The introductory reading is short, but it uses many words students may not know. You may want to have them read it out loud in class, discussing new words as they come up.

Day	Discussion	Homework Suggestions
Day 1	Do the introductory reading and work on Problems 1–3.	Problems 4, 5, and 7
Day 2	Work on Problem 6. Hold class discussion as suggested in Student Module.	

What's coming up? This discussion helps set the stage for students' later work, when they will give fully specified directions in Logo.

You might ask additional questions in the class discussion, such as:

- In the tourist map of Seattle, you can see that 1st Ave. bends to the left. Why might that bend not be mentioned in directions for a person driving the route?
- Why do the directions not give turning angles and exact distances for driving?
- Why were the maps we drew in class not identical?

ASSESSMENT AND HOMEWORK IDEAS.......

Homework is suggested in “Teaching the Investigation”. This is a lead-in activity, so formal assessments will come in later investigations.

ALGORITHMIC THINKING: DIRECTIONS FOR ROBOTS

Technology: Logo (See "Additional Resources" for information on availability of a free version.)

OVERVIEW

This is the first of several investigations using a formal language for specifying geometric shapes. Formal languages (like algebra, or in this case Logo) allow students to express their mathematical ideas in ways that can more easily be analyzed. This investigation introduces useful commands in the context of maps and writing directions. Subsequent investigations focus on the analysis.

The main ideas:

- developing and writing algorithms, thinking about geometric figures in an algorithmic way
- describing geometric shapes by stating rules for forming them (rather than by attributes of the fully formed shapes)
- Logo commands: `forward 37`, `right 72`, `cs` (or `cg`), `back -50`, and writing procedures

Students need no prior knowledge of Logo, though comfort with the computer is helpful. Even the teacher needs no fluency with Logo commands, but someone in the class—the teacher or a student—must know where to type a procedure definition that is to be saved to a file, how to define and save such procedures, and where to type direct commands.

The day before: Do you need to reserve a computer lab? Make sure Logo is installed on the computers. If you will be using the Supplementary Logo Activities, type in the maze procedures and have them ready for students.

This is best read out loud in class. Students can "act out" the Logo directions from the `PikeToNeedle` procedure in front of the class.

TEACHING THE INVESTIGATION

Day	Discussion	Homework Suggestions
Day 1	Students read the first page of the investigation and work on Problems 1 and 2. Discussion is suggested in the Student Module.	Problem 3 (Write out the procedure so it can by typed in and tested the next day.)
Day 2	Type in and test out Problem 3. Work on additional Logo activities (see below).	Problems 4, 6, and 7
Day 3	(optional) Discuss homework. Work on Problems 5 and 8. Include one or two additional problems designed by the teacher, if you want.	

Students who have never used Logo before may especially need help with the mechanics of typing: navigating between where you write procedures, where to type commands, where to put and not put spaces, and so on. Working in pairs may be useful, but more than two at the computer is probably not productive.

The Student Module covers all the Logo you need to know to solve the problems. If your version of Logo comes with a manual, all you need to find out is where to type in programs (where is the editor?) and how to run them.

Problem 1: Generally, commands that are part of a procedure are typed in a separate place from commands that are to be executed right away. In LogoWriter and MicroWorlds Logo, they are typed on the "back of the page"; in many Logos, they are typed in a separate editor. In UCB Logo, students may call the editor with `edit`, type there, and then close that window to go back to the "command center," or they may type their procedure to the command center directly.

Problem 2: You may wish to use this problem as an opportunity to point out the importance of careful definition. *Both* 45° and 135° are reasonable answers to "What is the angle. . . ?" but it is potentially dangerous to leave the communication so loose. (Imagine asking people to build some object with a 45° angle in it, and them not knowing whether that was to *look like* 45°, or to involve 45° of *turning*!)

Some students like direct, immediate feedback from the computer.

Problem 5: Some students find it easier to draw the map first with paper and pencil, then write down the procedure, and only type it in the computer at the end. Having students work in pairs and check each others' procedures by following the directions themselves may help avoid frustration.

Problem 8: This question may seem merely to be about programming, but it is equally a mathematical issue. In writing an algebraic expression, numbers cannot roam free without operations that use them, nor can operations be performed without the correct number of inputs to satisfy them.

ASSESSMENT AND HOMEWORK IDEAS. . . . . . .

Some specific homework assignments are suggested in "Teaching the Investigation" above.

A general homework strategy when using Logo is to have students write out procedures at home and be ready to type them in and test them out at school. You can check for existence of a written procedure before they begin work on a computer. Also, you can give students procedures you create and ask them to "be the turtle," drawing what the procedure should produce.

For assessment:

- Problem 4, if not used for homework, is a good assessment of students' ability to read a Logo algorithm. Similar problems are easy to create yourself.
- Other problems that ask "Which of these procedures create the same picture?" make good assessments. Give students four procedures like the ones below, and ask which of the pictures will look the same. (In this case, they should worry about only the *shape* and not its *position* on the screen.)

```
to Pict1  cs          to Pict2
 fd 100                 cs
 bk 100                 fd 100
 rt 90                  rt 90
 fd 100                 bk 100
 bk 100                 rt 90
 rt 90                  fd 100
end                     rt 90
                        bk 100
                        rt 90
                      end
```

```
to Pict3  cs          to Pict4
 fd 100                 cs
 rt 90                  rt 90
 fd 100                 fd 100
 rt 90                  rt 90
 fd 100                 fd 100
 rt 90                  rt 90
 fd 100                 fd 100
 rt 90                  rt 90
end                     fd 100
                      end
```

WITHOUT TECHNOLOGY . . . .

If you have only one computer available for demonstration, you can still work through most of this investigation. Students work in groups or individually to create the procedures; then a few of them are typed in and tested out for each problem. Make sure that different students' procedures are tested for different problems.

If no computers are available, you can work through the problems by behaving like the turtle yourself (or asking students to do so) at the board. With chalk, they follow exactly what they are told to do without any embellishment or assumptions.

SUPPLEMENTARY LOGO ACTIVITIES . . . .

The student materials use street directions to introduce Turtle language. There are many good alternative ways:

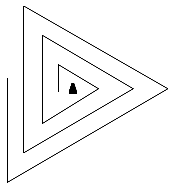
Many teachers introduce Logo by having students "act out" forward (fd), back (bk), right (rt), and left (lt) by giving directions to each other for a simple path like a rectangle.

If you know how to create Startup files in your version of Logo, then the maze procedures can be "waiting" for the students when Logo starts up rather than having students run them. You can look in your Logo manual for more about this.

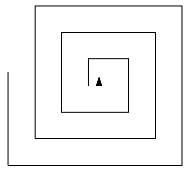
In their early experience on the computer, students may drive their turtles out of mazes. This is a nice thinking problem, with geometric payoff. Tell students to open the file with these procedures and run them; then they type Maze1 to begin. They can use fd, bk, rt, and lt to escape from the maze. Then try the other mazes (Maze2, and so on) in any order.

```
to Maze1
 cs Spimaze 3 40 30 9
end

to Maze2
 cs Spimaze 4 40 20 12
end

to Maze3
 cs Spimaze 5 40 10 15
end

to Maze4
 cs Spimaze 6 40 6 18
end

to SpiMaze :n :dist :inc :times
 if :times = 0 [fd :dist / 2 finishup stop]
 fd :dist rt (360 / :n)
 spimaze :n :dist + :inc :inc :times - 1
end

to FinishUp
 pu home rt 90 fd 20 lt 90 pd
end
```

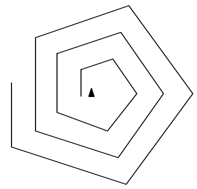
ALGORITHMIC THINKING: DIRECTIONS FOR ROBOTS (continued)



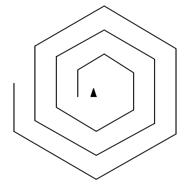
Maze1



Maze2



Maze3



Maze4

Students also enjoy driving the turtle around to draw their initials or other pictures they might invent. For such student-initiated projects, they will also need to know how to clear the screen using `clearscreen` (`cs`). In LogoWriter or MicroWorlds, this is `cleargraphics` (`cg`). They may also need the turtle to move without drawing. `penup` (`pu`) "retracts the pen" so that subsequent `forward`s do not leave a trace; `pendown` (`pd`) "extends the pen" so that subsequent `forward`s draw. Each Logo has its own way of handling pen color, erasing single lines, possibly filling shapes, and so on, if these features are desired.

For the current purposes, developing fluent use of a few commands is far more important than developing less fluent use of a greater number of commands.

ADDITIONAL RESOURCES ....

Brian Harvey (2634 Virginia St., Berkeley, CA 94709) of the University of California at Berkeley has developed a sophisticated but simple-to-use version of the Logo language with the intention of giving it *free* to educators and other interested people. He writes: "If you're on the Internet, you can download the Unix, DOS, or Mac version from anarres.cs.berkeley.edu in the directory pub/ucblogo." (If you're on the Internet and don't know how to download by anonymous FTP, ask.) You can also download Logo from Brian's Web page:
http://www.cs.berkeley.edu/ bh/

Robot turtles exist and may be of interest and motivational value to many students. Harvard Associates, Inc. produces a robot named "Roamer."

Harvard Associates, Inc.	(617) 492-0660
10 Holworthy St.	(617) 492-4610 (fax)
Cambridge, MA 02138	pclogo@harvassoc.com

MATHEMATICS CONNECTIONS

The following Logo "primer" may be helpful for your class:

Logo glossary:

fd	forward
bk	backward
rt	right turn
lt	left turn
pu	pen up (lifts turtle's pen)
pd	pen down (lowers turtle's pen)
home	Return turtle to home position.
cs	clear screen
or cg	clear graphics

(depending on version of Logo)

Other things to know:

- Logo, except on very old computers, is not case sensitive, e.g., Fd = fd.
- Different versions of Logo have different ways to edit and save files.

Geometry studies all kinds of shapes, including the complex and beautiful shapes of nature and the graceful curves of figure-skating. One way of dealing with the complexity of curves and curved surfaces is to treat them as if they were patched together from very small straight pieces with very large angles (small turn angles) between them. The branch of geometry that looks at shapes this way is known as *differential geometry*.

This "crawling along the edge of the figure" view is extremely valuable for understanding the figures of Euclidean geometry as well. Where differential geometry uses infinitesimal crawls and turns, "difference geometry" uses measurable distances and angles.

Difference geometry is also called "turtle geometry." Familiar geometric descriptions tend to focus on attributes of a completed figure. For example, a square is a polygon with four equal sides and four right angles, or a square is the union of the four segments connecting the four points $(0, 0)$, $(a, 0)$, (a, a), and $(0, a)$—or more generally, (x, y), $(x + a, y)$, $(x + a, y + a)$, $(x, y + a)$—in sequence (and then the last to the first).

The different points of view lead to different insights. To the Euclidean geometer, a

circle is a curve of constant distance from some special point (the center). A line is quite different. To the differential geometer, the circle is a curve of constant curvature (its amount of curve is the same everywhere). A smaller circle is more curved than a larger circle (we must bend a wire more to make a little circle than to make a big one). The larger a circle, the less curved it is, though, like all circles, its curvature is everywhere the same. Remarkably, a line is also a curve of constant (zero) curvature. Therefore, for the differential geometer, lines and circles belong to one family. You can get a sense of this by constructing a circle through three points (the circumcircle of a triangle) in The Geometer's Sketchpad® or Cabri Geometry II™. If one of the three points, say A, is moved slowly from one side to the other side of the line through B and C, the circle passes from "curved one way" to "straight" to "curved the other way."

This investigation and the three that follow develop parts of the difference-geometry perspective and use them to help students analyze important geometric ideas. The principal idea is the total amount of turning around a point (the 360° sum of exterior angles of a polygon).

Teaching Notes

Investigation **1.10**

Student Pages 69–76

ALGORITHMIC THINKING: ANGLES AROUND A CENTER

Technology: Logo

What's coming up? Students will later learn how to add variables to their procedures, making them work for *any* scale they want, introducing the idea of similarity and connecting it with multiplication.

The day before: Do you need to reserve the computer lab?

This introductory reading is very short. By now, students should be able to read it alone for homework the night before or in small groups in class.

OVERVIEW

In this investigation, students are introduced to more advanced programming ideas that allow them to structure their algorithms. The students may have already expressed some frustration over the tedious and long programs they were writing up to now and are ready to move on.

The main ideas:

- Measuring angles around a single point and determining exterior angle sum in polygons
- Developing and writing algorithms; thinking about geometric figures in an algorithmic way
- Describing geometric shapes by stating rules for forming them (rather than attributes of the fully formed shapes)
- Structuring these descriptions in ways that clarify certain properties
- Exploring the Total Turtle Turning (TTT) Theorem—the foundation for ideas about exterior and interior angles in polygons

Students need to know the Logo covered in Investigation 1.9.

TEACHING THE INVESTIGATION

Here is one possible teaching sequence. Depending on how long your students take to create these procedures and how much class time you want to spend discussing different procedures or homework, it may take more or less time for your class.

Day	Discussion	Homework Suggestions
Day 1	Read the introduction to the investigation and work on Problems 1 and 2.	Finish Problem 2 and do Problem 3.
Day 2	Problems 4–6	Problems 7, 9–11
Day 3	Test out answers to Problem 7 and correct them if necessary. Work on Problem 8 and Problems 15–17.	Problem 18

Day 4 Problems 12–14	Test procedures for Problem 14 and revise them if necessary.

If students are using experimentation, they may check their results by asking the turtle to make one more spine at the very end and seeing if this final spine coincides with the first.

Problem 1: The major task is to figure out exactly how much the turtle must turn between spines. Once students get that, the procedures are fairly straightforward. Using a protractor to measure the angles on the paper, reasoning about it using their knowledge about angles, and making rough approximations and experimenting with the software are all acceptable techniques.

Problem 2: This is a good time to talk about experiment versus theory. Experiments give data, which may lead to a theory. More experiments can check the theory, but theories are also used to "check" experiments. The theory, if it seems to make enough sense (or is subjected to proof, making it a theorem), can be used to "clean up" errant data to handle cases for which experiment alone may not feel accurate enough, or to bring precision to numbers that experimentation can only give as approximations.

$\frac{1}{1/a} = a$ is another example.

Problem 13: This is not so much about multiplication of positive and negative numbers as about inverses of inverses. $-(-a) = a$ is just one example (but may be the most familiar one). If students know about other inverse operations, they should be making those connections also.

Problem 16: At this point, students may say things like the following: "A turtle that goes for a trip and returns to its original position will eventually have turned a total amount of 360°." This is not yet a precise statement, and may be refined in the next two experiments.

Problem 17: Students may make a list with partial conjectures, including statements like the following:

> "When the turtle crosses its path the total turning is different."
>
> "The total turning is either 0° or 720°."
>
> "When the turtle turns clockwise it turns 360, and when it turns counterclockwise it loses 360."

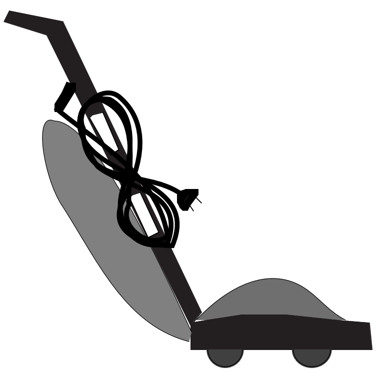
Their new theorem statement will depend on the kinds of observations they make. They may improve the original simply by qualifying it: "as long as the turtle's path doesn't cross itself . . . "

Don't forget the general homework strategy for activities involving Logo: students write out procedures for homework so that they are ready to type them in when they get to class.

ASSESSMENT AND HOMEWORK IDEAS.......

- Problems 10 and 11 are good assessment problems. If you want to give them as homework problems, you can create similar problems for an in-class assessment activity.
- If you use the "Take It Further" to focus on the Total Turtle Turning Theorem, then Problem 18 is a good final assessment. If students have worked in groups, Problem 18 could be given for homework or worked on in class to build in individual accountability.

WITHOUT TECHNOLOGY ....



Here is an experiment that can help you and your students visualize the Total Turtle Turning Theorem: Bring in an electrical cord (like an extension cord) and lay it out in a straight line. There are two different ways you might wind it up to put it away. You might loop it around your arm, or you might wind it in a figure eight (like a vacuum cleaner cord).

First wind it in a loop around your arm. You can probably feel the cord twisting as you loop it. When you're done, rather than unwinding it, simply grab the two ends and pull the cord out straight. The cord will have twists in it (about one for each loop).

Straighten the cord and start again. This time wind it in a figure eight (you might need someone to help you hold it). When you are done, again grab the two ends and pull the cord out straight again. There shouldn't be any twists in the cord.

The situation is the same when the turtle crosses its path more than once: all the turning can be in the same direction, or some of it can be in the opposite direction. If the turtle ends up back where it started, though, the total turning is always a multiple of 360°. Ask the students questions like the following: "How does the total turning angle relate to the amount of twisting?"

Teaching Notes

Investigation **1.11**

Student Pages 77–82

Algorithmic Thinking: Spines, Stars, and Polygons

Technology: Logo

Overview

This investigation uses computer programming to introduce powerful mathematical ideas including: generalizing, making one process that covers a variety of situations, and scaling size (but otherwise keeping an algorithm fixed) to get similar figures.

The main ideas:

- exterior and interior angles in polygons
- angle measure around a single point and exterior angle sum in polygons
- developing and writing algorithms; thinking about geometric figures in an algorithmic way
- describing geometric shapes by stating rules for forming them (rather than by attributes of the fully formed shapes)
- regular polygons
- Logo commands

Students need to know the Logo covered in Investigations 1.9 and 1.10.

Regular polygons are discussed in this investigation, but no previous knowledge of them is required.

The day before: Do you need to reserve the computer lab?

Teaching the Investigations

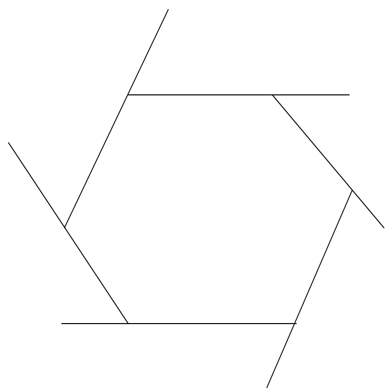
This plan uses part, but not all, of the “Take It Further” problems. You may want to use a different subset or none at all, depending on your class.

You may want to do this reading out loud in class so that you can discuss the new concept of using variables. If your students know algebra, you can relate the two uses of the word “variable.”

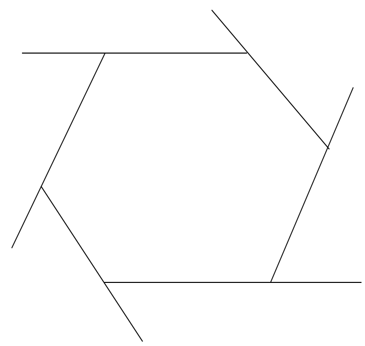
Day	Discussion	Homework Suggestions
Day 1	Read the introduction to the investigation and work on Problems 1–3.	Work on Problem 4 (have a procedure ready to type in and test out) and Problem 5.
Day 2	Test out the `BetterStar` procedure (Problem 4) and revise it if necessary. Problem 6; class discussion about Problem 7.	Problems 8–12

Day 3	Discuss homework (especially Problem 12). Work on Problems 15 and 16.	Problems 13 and 14 and the "Write and Reflect" (Problem 20)

Note that the sum of the exterior angles is different from the turtle's "total turning" in polygons. The turtle turns in only one direction and therefore covers only half of the exterior angles. (So the sum of the exterior angles in a polygon would be 720°.)



These sum to 360°,



and so do these.

Problem 16: The instructions to use Star and Shape are intended to encourage students to analyze the pictures in terms of shapes they already know, but students are not restricted to these two procedures; they may use *any* Logo commands they know.

The pentagonal web is harder than the hexagonal. Because the triangular parts in the pentagonal web are not equilateral, designing an algorithm for drawing this picture involves solving many new problems and requires a great deal of trial and error, at least initially. Students might find it interesting to explain why these two figures that look so much the same are not equally easy to draw.

Students have the needed facts to figure out the angles in the PentaWeb but may not realize that they do. Each fifth of the web is a nest of triangles with a common vertex

at the center of the five-spined star. The measure of the central angle is $\frac{360^\circ}{5} = 72^\circ$, so the remaining two angles in each triangle must each be 54° (because 54 + 54 + 72 = 180). Therefore, the turtle must turn $(180 - 54)^\circ = 126^\circ$.

If students adapt the HexaWeb algorithm by changing only the angle and the number of spines (sides) in the figure, they can produce a figure that is fairly close, but the pentagons are slightly too small.

This version is not quite right!

```
To PentaWeb
 vstar 5 80
 fd 10
 rt 126   vpoly 5 10   lt 126
 fd 10
 rt 126   vpoly 5 20   lt 126
 fd 10
 rt 126   vpoly 5 30   lt 126
 fd 10
 rt 126   vpoly 5 40   lt 126
end
```

Without trigonometry, students do *not* have the mathematical tools to calculate the proper size adjustment, but they can come very close very quickly through experimentation. To make such experimenting easier, it helps to build an algorithm with a "fudge factor."

```
To PentaWeb :fudge
 vstar 5 80
 fd 10
 rt 126   vpoly 5 10 * :fudge   lt 126
 fd 10
 rt 126   vpoly 5 20 * :fudge   lt 126
 fd 10
 rt 126   vpoly 5 30 * :fudge   lt 126
 fd 10
 rt 126   vpoly 5 40 * :fudge   lt 126
end
```

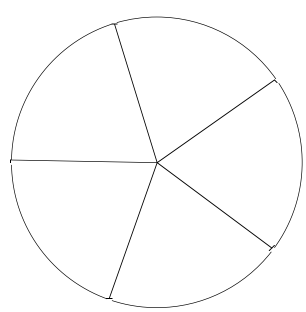
In just a few trials, they can zero in on the "fudge factor," which is about 1.18. This kind of design may be interesting to return to when students can use trigonometric arguments to compute the side to be 2 * sin 36.

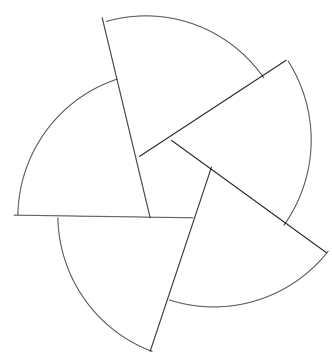
ASSESSMENT AND HOMEWORK IDEAS.......

- Problem 4 asks students to remember the last investigation and apply a result to solve a new problem. This is a good assessment of whether they connect their current and previous work.
- Problems like Problem 5 are good assessments. You can come up with other examples of two similar procedures for students to compare.
- Problem 12 asks students to describe a general rule. This, together with problems like Problem 11, which ask them to apply the rule, could form a quiz.

WITHOUT TECHNOLOGY ....

The example of the exploded star can help students see that the turtle uses exterior angles. Show the star "exploding" little by little until it gets the polygon shape. One teacher who wanted to do this but didn't have access to a computer lab decided to use construction paper arcs. By moving them away from the center, you can see the polygon emerge in the middle.





Along similar lines, you can ask students to do this thought experiment: Draw a polygon on your paper with each side extended in one direction as in the exploded polygon. Now imagine looking at that drawing from very far away, maybe from the moon. You see a dot in the middle with spokes coming out—it looks just like what you've drawn with the Star procedures.

Additional questions for discussion or written homework:

1. How much turning makes a complete revolution? (How many degrees does the turtle turn to end up back where it started?)

2. How much turning would make half of a revolution (exactly reversing the turtle's direction)?

a. The phrases "three sixty" and "one eighty" are used in English to describe many situations of full and half turns. Describe a specific usage of one of these terms (when you've heard it, seen it, or used it yourself).

b. In drawing `BetterStar 5`, the turtle never drew a line in precisely the reverse direction as any other line. In which stars *does* it draw pairs of lines in exactly opposite directions?

ADDITIONAL RESOURCES . . . .

Abelson and diSessa, *Turtle Geometry*. Cambridge: MIT Press, 1980.

MATHEMATICS CONNECTIONS . . . .

Compare Logo procedures with variables to defining a function algebraically. What the function produces depends on what number it receives for its variable. To define a polynomial function, one might write $f(x) = x^2 + 3$. To apply the function, one writes $f(2)$ or $f(m)$, but the x is no longer important. Similarly, when one defines a procedure in Logo, one gives names to any required variables . . .

```
to Rectangle :h :w
 repeat 2 [fd :h  rt 90   fd :w  rt 90]
end
```

. . . but when one *uses* the procedure one types `rectangle 20 40` or `rectangle 50 (50 + 50*sqrt 5)`, and does not make any use of the `:h` or `:w` unless they have independent meaning *outside* of the procedure.

ALGORITHMIC THINKING: SPINES, STARS, AND POLYGONS (continued)

Problem 19: This *is* an extended project that may involve some students in thinking about common divisors. Tentative and partial conjectures are worth noting, and coming back to.

Another Logo project dealing with lengths and angles is the classic problem from Abelson and diSessa (see "Additional Resources") that has students analyze what seems to be a simple Logo procedure:

```
to Inspi :s :a :i
  fd :s rt :a
  inspi :s :a+:i :i
end
```

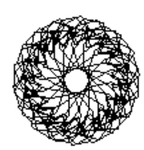
Inspi takes three inputs: *s* determines the size of the thing; the shape is determined by *a* and *i*.

Sometimes the shape is "back and forth" (the numbers under the picture show the inputs),

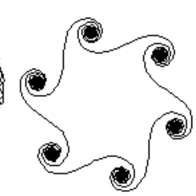
10 20 5

5 20 17

and sometimes the shape is like a polygon,



15 20 18



5 20 3

1 17 2

and sometimes it just marches away, never to be seen again.

The object of the game is to predict the shape from the inputs. This is a great arena for conjecturing, experimenting, analyzing, and proving. There's a beautiful interplay between geometry and arithmetic. For example, when will you start to repeat your steps? This happens when you're at iteration k, where k is the smallest integer so that ka is a multiple of 360°. When does that happen? It happens when

$$k = \frac{360}{\gcd(a, 360)}.$$

And where are you pointing when this happens? Your heading is the remainder when

$$a + (a + i) + (a + 2i) + \cdots + (a + ki)$$

is divided by 360, and that's a nice thing to add up.

The connections to other mathematics, especially number theory and arithmetic series, are beautiful. The geometry requires one to think about total turning, invariant states, and other ideas from differential geometry and topology.

On the habits-of-mind front, we can make experimental design and heuristics an important part of the investigation. Why does s not matter? Why should you keep one variable constant and vary the other? *Should* you keep one variable constant and vary the other? Should you study the behavior of this system by experimentally varying parameters, or should you spend some time analyzing the algorithm that produces the pictures? This is a chance to hammer at the the big message: There's a difference between data-driven insight and the insight you get by the analysis of algorithms. Both are satisfying, but they're different.

One of our students, who was barely passing algebra, took the better part of a semester to fully understand what was going on. We ran into him recently. It's been 10 years since he's been in high school, but he said he still has his notebook from this project.

Another pair of very-advanced students took this on and developed all of the number theory to analyze the problem in a couple of weeks.

A possible extension to Inspi:

```
to Spi :s :a :i
  fd :s rt :a
  spi :s+:i :a :i
end
```

Algorithmic Thinking: Irregular Figures

Technology: Logo

Overview

This short investigation asks students to pull together what they've learned about Logo to create some irregular shapes.

The main ideas:

- recognizing and analyzing geometric attributes of fully-formed shapes in order to develop an algorithm for constructing them
- describing geometric shapes by stating rules for forming them (rather than by attributes of the fully-formed shapes)
- developing and writing algorithms; thinking about geometric figures in an algorithmic way
- angle measure around a single point and exterior angle sum in polygons
- exterior and interior angles in polygons
- sum of interior angles in a triangle is 180°
- using the Total Turtle Turning (TTT) Theorem

Students need to know the Logo covered in Investigations 1.9–1.11.

The day before: Do you need to reserve the computer lab?

Teaching the Investigation

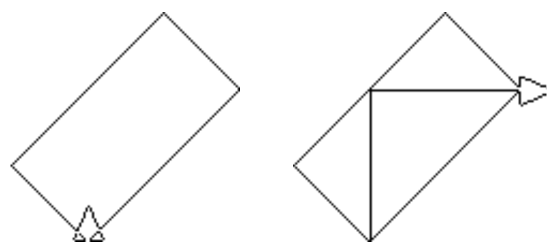
This is a very short investigation; here is a two-day teaching plan:

Day	Discussion	Homework Suggestions
Day 1	Work on Problems 1–4.	Problems 5 and 6
Day 2	Problem 7: students complete as many procedures as they can.	

Problems 1 and 2 are essentially the same: Problem 1 poses the task, and Problem 2 adds some structure to help students solve it. In Problem 16 of the previous investigation, students could figure out the angles for the webs, but had to experiment to determine the lengths. In the current problem, even the angle remains unknown without trigonometry or experimentation. Students who know the Pythagorean Theorem can use it to find out the required length, but many students will need to experiment for both the length *and* the angle.

Some students may know that the `home` command does what they want here, but it does not allow them to answer Problem 3.

Problem 7: No new Logo or mathematics is needed to draw these pictures, but some experimenting and ingenuity may be needed. Analyzing the pictures before plunging into drawing is very helpful. For example, it simplifies one's work greatly if one notices the relationship between the following two pictures:



Also, the fact that the "decoration" inside the second figure appears to be two sides of a square constrains the shape of the surrounding rectangle: the only suitable rectangle will have sides s and $2s$, and the two internal lines must each be of length $s\sqrt{2}$.

ASSESSMENT AND HOMEWORK IDEAS. . . . . . .

- Problem 4 asks students to remember previous work and relate it to a somewhat new situation. This is a good assessment of the content of the previous investigation.
- Problem 6 is useful as an assessment because it asks students to integrate the reasoning that they have developed throughout this investigation.
- Problem 7, despite its seemingly arbitrary designs, takes advantage of knowledge developed earlier and may also serve assessment purposes.

Constructing from Features: Moving Pictures

Technology: Geometry software (See "Getting Started with Geometry Software" in these notes.)

Overview

Students learn the basics of geometry software while exploring the features necessary to determine some familiar shapes. Geometry software constructions can be directly compared to constructions on paper with ruler and compass and can often use the same algorithms. However, the software usually has more tools and more correct strategies. It is the strategy that counts. In each case, students must look for crucial geometric characteristics of the requested figure.

The main ideas:

- essential properties of a rectangle, square, parallelogram, equilateral triangle, and rhombus
- exploration of geometry software
- introduction to thinking about similar figures

The day before: Do you need to reserve the computer lab? Is the software already installed on the machines?

If you want your students to get more practice, your software probably comes with tutorials. You could have students work through them before or after this investigation.

Students will learn about the software while working through this investigation; they don't need to know it already.

Students work through the problems in order, with occasional breaks for class discussions. The two main challenges are:

- keeping students on task and accountable for progress while working on the computer;
- assigning appropriate homework.

If you are just starting out using computers with your students, you will need to set some ground rules about progress and computer use. After you decide what reasonable progress is for that day, you will also want to decide how students will demonstrate this progress. Some options are:

- They might print their screen or sketch what they see on the screen at the end of each problem (or group of problems). On the same paper, they can write a paragraph about how they made the construction, anything they tried that didn't seem to work, and any conjectures they have.
- You can write questions on the board or on a transparency (or a worksheet to fill out) which depend on having constructed the sketch.

TEACHING THE INVESTIGATION ....

"Reasonable progress" for most classes working through this investigation would be something like this:

Day	Discussion	Homework Suggestions
Days 1 & 2	Work through Problem 4. There may a lot of start-up time involved at the beginning. You may want to begin with a whole-class reading and discussion of the introductory page so that they get the point of the software they will be using. There can be difficulty in starting up the software, accidentally closing windows, finding the right tools, and so on. After the first two days, however, students will be more comfortable and things will move more quickly.	Read the short piece on "Drawings *vs*. Constructions."
Day 3	The windmill construction: A class discussion about the difference may help them focus their work a bit more.	
Days 4 & 5	UnMessUpable figures and the scavenger hunt: It is helpful to wander the room and test as many of the constructions as you can during class, but you may not get to every one. Have students save their constructions to a disk with their name written on it, using informative file names like "rectangle," "square," and "T." It won't take much time for you to test the sketches after school or before school the next day. Let students know which ones didn't work, and ask them to correct those constructions as part of their second day's work.	
Days 5 & 6	One of the suggested projects	

Throughout this investigation, encourage comparisons to constructions made with compass, ruler, and protractor, and have the class come up with a list of the objects they would call "basic to geometry."

In the "Constructing a Windmill" section, there are a lot of steps and a lot of pieces to keep track of. If you need it, the manual for your software provides instructions on how to create intersection points, hide objects, and so on.

What's coming up? Having radius defined by $\overline{BC}$ doesn't mean segment $\overline{BC}$ is *attached* to the circle—it just means that $\overline{BC}$ determines the *size* of the circle. Many future investigations will require students to copy lengths in this way.

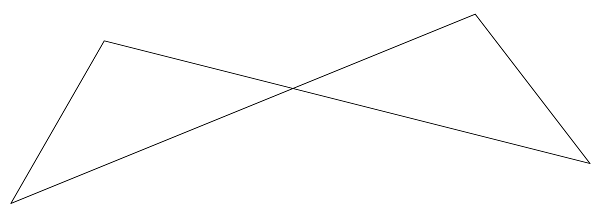
The windmill exercise covers all the methods (and more) necessary for the next section, but the students will probably need reminding about the importance of *constructed* figures and the way to hide parts of a construction.

Problem 6 is an opportunity to make connections with what some students have learned about graphing lines in algebra. In particular, the existence of a "point-slope form" for the equation of a line says that these two features—a point and a slope (parallel or perpendicular to a given line)—are enough to determine a line uniquely.

For Problem 10, you may want to remind students of what they learned in Problems 11–13 in Investigation 1.1.

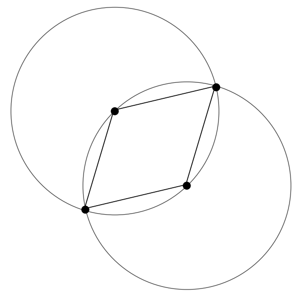
In plane geometry this is not generally considered a quadrilateral: Polygons are conventionally defined to prevent self-intersection, and quadrilaterals are 4-sided polygons. In a branch of geometry known as *projective geometry*, however, this self-intersecting figure *is* considered a quadilateral.

The sidenote to Problem 18 asks about creating quadrilaterals with both pairs of opposite sides equal in length but not parallel. In fact, there is no such "ordinary quadrilateral," but there are other 4-sided figures with this property. In a "cross-quadrilateral," for example, this is possible. The diagonals and two nonparallel sides of an isosceles trapezoid form such a cross-quadrilateral.

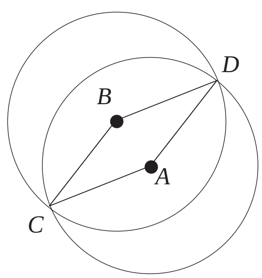


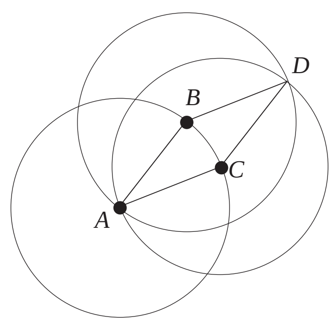
Problems 20–21: Students have already used most of these tools, but some new ones are introduced. The rotation and reflection transformations, avoided earlier, are discovered now. Some classes have used this as an opportunity to discuss, systematize, and consolidate what was learned. Some students, after performing all the tasks, divided up the job of writing up their results in the form of a "manual" of geometry software tools and their uses.

For Problem 23, students may not know what a rhombus is yet. When they are asked to construct one, they'll *want* to know, so that's a good time to tell them (or to have them tell each other). Some students may create a rhombus by connecting the intersection points and centers of two congruent circles, like the following figure:



Such a rhombus will always have angles of 120° and 60°. Count this as a real solution, but challenge them to make a rhombus that is less limited. For example, observe the two-circle and three-circle solutions below:





ASSESSMENT AND HOMEWORK IDEAS.......

The following problems from the Student Module don't require use of the software and could be assigned for homework as well:

- Problem 6 (for the night after students complete the windmill construction)
- The sidenote to Problem 10
- The sidenote to Problem 14

- The “Write and Reflect,” after students complete the “Drawing UnMessUpable Figures” section
- “Checkpoint Problems” 22 and 24

Homework for computer-based investigations is a challenge. One idea is to ask students to reflect (in writing) on what they did in class:

- Describe two things you learned about the software and two things you learned about geometry.
- Make one or two of the same constructions using hand tools (straightedge and compass). Describe your method and compare it to the computer-based method.

The methods that students use for solving these problems are often very rich in information about how students are thinking and what they know. Ask them to describe their methods and why they work. The projects in the “Take it Further” section can serve as final assessments. Also, many of the investigations in the next section of the module use geometry software in the search for geometric invariants, so you can wait and use some of *those* problems to assess students’ understanding of construction techniques.

Getting Started with Geometry Software

When your students first get on the computer, they may need help. Try to determine which students can act as resources for others, and make it clear that helping each other is part of the process. You will want to roam and troubleshoot, but you can’t be everywhere.

Even though each “brand” of geometry software has a great deal of richness built into it, the basics (all you or your students will need for this module) are fairly easy to get, especially if you are already moderately comfortable with your computer and are willing to poke around and explore. Any drawing program must provide tools that assist in drawing the most basic shapes one might use, but because geometry software programs are designed for mathematical investigation and not just for drawing, they also provide ways of specifying relationships among the shapes, transforming, measuring, labeling, and modifying the appearance of shapes (including hiding construction lines).

Students will need to learn how to draw points, segments, rays, lines, and circles; to specify fundamental relationships like parallel, perpendicular, and radius equal to

Units and Precision: **Most tools provide a way to specify units of measure (centimeters vs. inches or degrees vs. radians) and level of precision.**

given length; to measure length, area, and angle; to hide construction lines; to label objects; to trace objects as they move; and to save their work. Later, they will also need to know how to perform simple calculations—sums, differences, products, and ratios—on measurements they take.

With this software, any geometric object may be specified in more than one way:

- A *point* may be placed arbitrarily, may be constrained to reside on an already-existing line or circle (or other object), or may be restricted even further to be the intersection of two objects.
- A *line* might be specified by two points or by a single point and a slope determined by some other object (parallel or perpendicular to another line).
- A *circle* may be specified by locating its center and a point on its circumference, or it may be constructed with a radius set equal to the length of some segment.

Make sure students do not delete objects that they have used to specify the characteristics of other objects. When a line is used to create a perpendicular or as the basis for a segment, but the final sketch will not show the original line, many students are tempted to delete the line as "unnecessary" to the the final sketch. *Hiding* the line preserves the necessary relationship, but *deleting* the line will free up the objects based upon it, or may even cause them to disappear.

Length, area, and angle measure are always provided, along with simple calculations on these values. Some software provides other measures and more advanced calculations (for example, slope, arc length, and trigonometric functions), ways to represent values in tables or graphs, and analytic geometry tools including coordinates and equations.

The ability to drag a point and watch the entire construction respond dynamically to that change is the feature that defines geometry software. What can be dragged varies greatly with the various software packages available.

As objects move, they can leave a trace of where they've been. This is often useful in analyzing the behavior of a construction.

Geometry software also allows you to add and may allow you to change the appearance (thickness and color, for example) of objects to help make a diagram more understandable or attractive.

Which software features to use and which to avoid: While there are genuine *mathematical* advantages to learning the software, there are also features of the software that,

at certain stages in students' learning, may be more distracting than advantageous. In particular, we feel that for the problems in this section, it is best *not* to make use of the transformational tools (rotation, translation, dilation) your software provides. However, introducing these tools may be appropriate if transformations have already been an important part of your students' prior study or if transformational ideas are to be taken up as a serious study with materials that supplement the problems in this module.

It has been our experience that introducing these transformations for the first time at this point tends to be a distraction from the geometry, and tends to be seen by the students as features of the software rather than as features of the mathematics. Also, the kinds of errors that students tend to make when using these tools for the first time are often difficult to explain in terms of the *mathematics*.

MATHEMATICS CONNECTIONS . . . .

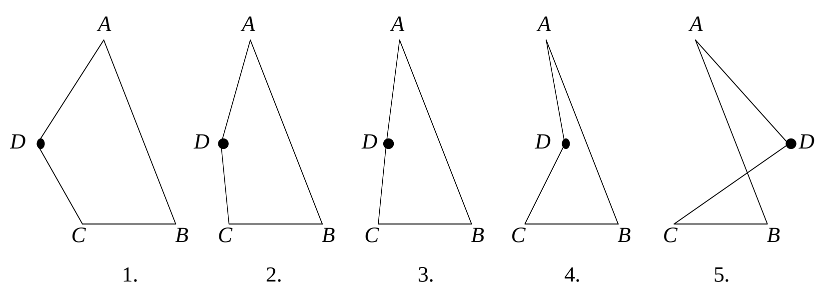
Sometimes in the hunt for invariants, it is relatively easy to perform experiments "in your head." Squares, for example, are so familiar and regular that a rough sketch is enough to focus your attention. But there are other times—when the objects to be studied are more complex than squares or come in far greater variety—when it is not so easy to perform experiments in your head. In these cases, models are useful for more than just focusing attention; experiments may be performed directly on the models. If the models are to be used in *this* way, they had better not be inaccurate in any important details; they must be built to contain the necessary features. It also helps if the model is *variable*—not just *one* triangle, but something that can give you insight into *all* triangles.

Consider how you might explore the properties of triangles. With pencil and paper, you can draw a single triangle and measure its angles and sidelengths. You would find that the angles sum to 180°. Is this an invariant for all triangles? You could start to check by drawing more triangles, but that would get tiresome quickly. Using geometry software, you only have to construct one triangle and instruct the software to sum its angle measures. As you drag one of its vertices or sides around, the sum of its angle measurements will be updated automatically, and you'll be free to examine as many different triangles as you'd like. While that is not a *proof*—you can't look at every possible example—it is strong confirmation that here is a phenomenon worthy of logical explanation or proof. Sometimes, such experiments give insights that help one find the proofs.

Problem 18: The solution provided in the Solutions Resource concludes with the mysterious statement: "Connecting the four points as shown does *most* of the job." The *rest* of the job may not be at all obvious, is debatably necessary if noticed, and cannot be done anyway! The problem lies in the definition of a quadrilateral. Plane geometry (but not some other geometries) generally restricts the meaning to a figure that does not intersect itself, but no geometry software makes it easy (or, as far as we know, even possible) to construct such a figure.

For certain classes, this may be a good opportunity to explore the *purpose* of definition (to help make communication reliable, to help make mathematical reasoning secure) and what particular restrictions students want to place within the definitions of "quadrilateral," "vertex," and "point." For example, by what arguments (or for what purposes) should all of the figures below be considered quadrilaterals, and by what arguments (or for what purposes) should quadrilaterals be restricted more narrowly? What definition would exclude cases 3 and 5 without excluding the others? What definition would include all of these but rule out such four-segment shapes as "M" or "#"?

Is D a "vertex" in case 3? Is it a "point"? What about the intersection of $\overline{AB}$ and $\overline{CD}$ in case 5?



These are not issues that are settled by mathematics but by one's purpose within the mathematics. In plane geometry, cases 3 and 5 are generally *not* considered quadrilaterals, nor is D considered a vertex in case 3. Designating one (or more) of the points along the sides of a triangle as a new "vertex" would make the triangle into a quadrilateral (or a pentagon, or . . .). But graph theory considers D a vertex in all of these figures and does *not* consider the intersection of $\overline{AB}$ and $\overline{CD}$ in case 5 a vertex. Projective geometry considers all five of these cases to be quadrilaterals.

Problem 22: Some ideas that are important and possibly new include:

Construction *vs.* drawing A *construction* is a *method* of building that (if executed correctly) guarantees that the desired properties are built in. A *drawing*, however accurate it may be, is a picture that was created using a method that cannot, in general, guarantee the right properties.

Extending or narrowing definitions Students who know the definition of "perpendicular to" often overlook the fact that segments can be perpendicular to one another without touching. Similarly, radius is both a segment and a measure: a circle can have a radius of BC without touching segment $\overline{BC}$. "Quadrilateral," "vertex," and even "point" may have acquired new refinements to their definitions.

Minimal-defining/uniquely-determining properties What properties uniquely determine a line? (two points or a point and a slope) What properties uniquely determine a circle? (center and radius or center and point on circle) What properties define a rectangle? A square? A parallelogram? A quadrilateral?

Particular constructions These include square, rectangle, parallelogram, equilateral triangle, midpoints, and angle bisectors.

This problem and others like it are investigated in depth in the Connected Geometry module *Optimization*.

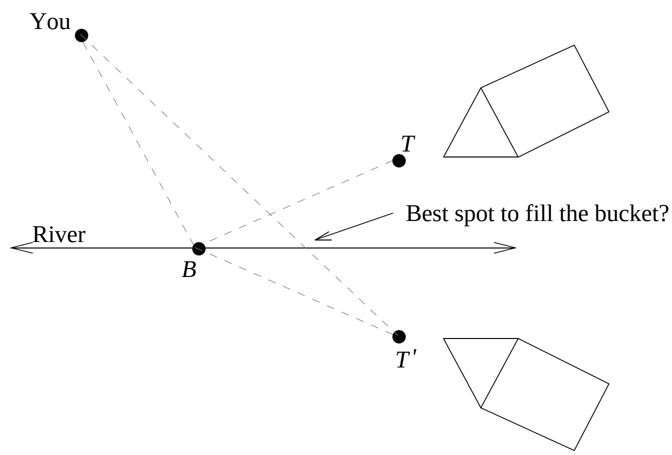
Problem 26: The Solution Resource does not give a solution, and, at this stage, the problem should probably be left open for discussion. Students have not been taught anything in a systematic way that would help them *solve* this problem, but do have a number of tools that they can use for investigating it. The three "Ways to Think About It" suggest useful approaches.

The "ballpark" approach suggests that the path shown in the Student Module is not the best. Most people would judge (correctly) that a spot along the river closer to the tent would be better.

If you consider the case of the tent right at the banks of the river, clearly the best spot to get the water is right near the tent. If *you* are right near the river and the tent is far away, the best spot to get the water is probably quite near you.

The "what do I know about shortest distances?" approach suggests a geometric insight.

In the sketch below, T' is the image of T, reflected over the line of the River, making BT equal to BT'.



Any bent-line path from You to some spot B along the River to T will be the same length as a path from You to B to T'. The shortest distance from You to T' is a straight line, so the best position for B will be where that straight line crosses the River.

WARM-UPS

Technology: Geometry software

OVERVIEW

The focus of this section of the module is on geometric invariants. We want students to look for things that stay the same (shapes, relationships, measures, and ratios) when everything around them is changing.

Students are already familiar with several mathematical invariants:

- the essential properties of squares, rectangles, parallelograms, and so on, that they used in constructions;
- the triangle inequality;
- the (interior) angle sum in a triangle on a plane;
- the Total Turtle Turning Theorem.

In this investigation, students will explore other numerical and geometric invariants. Geometry software allows students to change things in a continuous way and to watch what happens. One construction allows a student to test millions of cases to see if an invariant seems to hold. If you watch closely, you may get clues about *why* an invariant holds.

If you don't have geometry software, some of the investigations in this section of the module will be difficult to do in your class, but others can be adapted for manipulatives or paper-and-pencil exploration. See "Without Technology".

The geometric content in this part includes: parallel lines and angle relationships, perimeter and area, circles, ellipses, perpendicular bisectors, sums of angles in polygons, and concurrences in triangles.

The six problems in this section are "warm-ups" in numerical and geometric contexts. They provide a concentrated focus on looking for invariants.

What's coming up? π **will be explored further in the modules** ***The Cutting Edge*** **and** ***A Matter of Scale*****.**

The main ideas:

- invariants over a set; invariants as things vary; variety of *kinds* of invariants
- the relationship between the circumference and diameter of a circle
- some facts and ideas about parallel lines and angles

WARM-UPS (continued)

The day before: Do you need to reserve the computer lab?

Students will need familiarity with geometry software, based on the Investigation 1.13. They will also need the following vocabulary from the section "Geometric Languages and Tools.": *circle*, *diameter*, *circumference*, *parallel*.

TEACHING THE INVESTIGATION

Problem 1 can be done for homework the night before beginning the investigation and then discussed in class. Problem 2, if investigated on the computer, can pick up the continuing thread about ellipses, begun at the very beginning of this module. If no computer exploration of this problem is chosen, it can be done as homework with Problem 1.

Problems 2–6 can be split among pairs of students working together and discussed later, or they can be explored over two days of class.

Constant sums or ratios of measurements can be hard to spot because they require not just a measurement or two, but a calculation. Here is a useful way to think about looking for them:

WAYS TO THINK ABOUT IT

Two lengths that are both stretching or shrinking at the same time may be related in a way that does not change at all. They may have a constant difference—one may always be a fixed amount longer than the other. Or they may have a constant ratio—one may always be twice as long, or 79 times as long as the other. If one thing grows as another shrinks, perhaps their sum or product is constant.

Problem 4 asks students to find "invariants that aren't listed above." They may notice things like the invariance of EF, but feel that these are too trivial or obvious to be worth mentioning. While invariants such as these may not seem as interesting as the constant area, they *are* worthy observations. When students are looking for *proofs* of

"interesting" observed invariants, they should be in the habit of looking for other things that do not change (and may not appear at all interesting), upon which their observed value may depend. In this case, for example, the constant area is a consequence of the fact that *EF* does not change.

ASSESSMENT AND HOMEWORK IDEAS.......

These are just warm-up problems. Assessment will be provided in later investigations. If you take two or more days on this investigation, homework can be drawn from the "Searching for Patterns" section of Investigation 1.15.

WITHOUT TECHNOLOGY ....

Problem 3 can be done without the computer, with students creating and measuring their own circles and performing the measurements. (They will have to use string to measure the circumference.) Make sure that the class as a whole creates a variety of circle sizes and compares their results. The areas can be estimated by placing the circles on a square grid or graph paper.

MATHEMATICS CONNECTIONS ....

The discussion at the end of this investigation in the Student Module says

> "Even *outside* of mathematics, it is important to look not only for what changes, but also for what doesn't change. Finding a nontrivial invariant—something that otherwise different cases have in common—is often the key to a deeper understanding of a situation or phenomenon. Try to find examples from history, psychology, literature, or music."

This is a very deep idea, and finding really striking examples can make a powerful impression on students. It is good for them to find their own examples, but it is likely that they will need some help from you in order to get "unstuck" from thinking only about mathematical ideas. Here are a few suggestions to get *you* started:

WARM-UPS *(continued)*

Music and Art

- You can get to know the character of a piece of music by Bach or Beethoven, or of paintings by Picasso or Dali, well enough to recognize it in a *new* work that you've never encountered before, and therefore identify the creator. This thing we call "style" is an invariant—something that remains the same while all the details are different. In the same way, rock music, country and western, classical, and rap are all distinct categories because there is something about them that is recognizably "the same" despite the great variety within each category.

  Without expertise in music or art, we may have a hard time *articulating* what features these works have in common, but we can recognize them. Features like these become especially important when we are trying to identify newly-discovered works or settle disputes about authorship. They are also of value to people who are trying to emulate certain styles.

- Within a single musical composition, there is often a repeated theme—a pattern of rhythm or melody—that runs through the piece or recurs from time to time. One part of what the ear hears remains the same as the piece progresses in time. Visual patterns, as in Islamic tiling, also make use of invariants. One part of what the eye sees remains the same, even as one's gaze shifts locations within the pattern.

History

To make a coherent history, it is necessary to find some unchanging elements. In writing the history of France, for example, birth and death changes all the people, building and destruction changes the appearance, political changes take place, and even the language evolves considerably. What brings coherence is the sense that something did *not* change, something more than just the plot of land on which the story unfolds. More interesting invariants are the patterns people try hard to find to explain history—patterns that we might be doomed to repeat if we do not understand them well enough.

Social Sciences

In order to predict, we must find patterns. Whether we are trying to track down a criminal by looking for a pattern of behavior (the "modus operandi"), investing in the stock market, or trying to buy clothing that will suit the taste of a friend, we are always looking for patterns—things that are stable enough to predict—amidst all the uncertainties and variables.

WARM-UPS (continued)

Science

One group of animals, despite great variety within the group, has backbones, while another does not. In fact, any classification scheme is a way of highlighting what does *not* change.

Numerical Invariants

Technology: Geometry software

Overview

This investigation introduces students to the hunt for numerical invariants: constant measure, sum, product, ratio, and difference.

The main ideas:

- knowing when to look for constant sum or product (if two quantities have the same sign and change in opposite directions)
- knowing when to look for constant difference or ratio (if two quantities have the same sign and change in the same direction)
- 90° angle inscribed in a semicircle
- fixed segment as a constant length
- sums of angles in polygons
- power of a point (informal exploration)
- discovering that any line parallel to the base of a triangle cuts the other sides proportionally

Students must be familiar with geometry software from Investigations 1.13 and 1.14. They must also understand the term "invariant."

The day before: Do you need to reserve the computer lab?

Teaching the Investigation

Before starting the investigation, assign Problems 3 and 4 for homework.

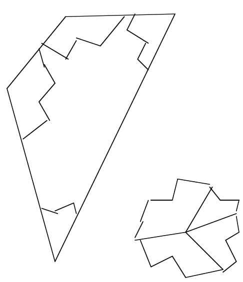
Here is one possible plan for working through the investigation:

Day	Discussion	Homework Suggestions
Day 1	Students have completed Problems 3 and 4 for homework; discuss in class. Students then work on Problems 1 and 2 and Problems 9 and 10. Discuss results.	Problems 6–8
Day 2	Students work on Problem 11 (on or off the computer) in groups. Then they present their results and how they found them to the class.	Problem 11

NUMERICAL INVARIANTS *(continued)*

Day 3	Problems 13–17	Write up results of the investigation.
Day 4	Problems 18–20	

The triangle paper-ripping activity from Investigation 1.5 can be extended to quadrilaterals, with the pieces completely surrounding the point where the corners match up:

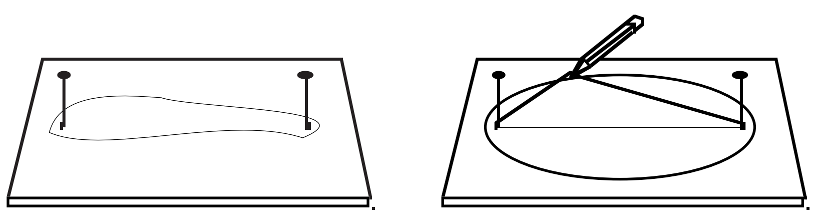


The text below Problem 10 suggests that trapezoids, parallelograms, rectangles, pentagons, and hexagons might each have their own special fixed number. Until there is some reason to believe that the critical variable is the *number* of sides rather than some other feature (like regularity or parallels), students should not assume that what is true of rectangles will necessarily be true of trapezoids.

This problem can be time-consuming and is well suited for group work. Students may divide up the tasks and share their findings in order to build a table and look for a pattern. Some students may prefer working with geometry software; others may prefer paper-cutting or drawing-and-measuring approaches.

Problem 20: The third locus is an ellipse. At this stage, few students will be familiar with ellipses, let alone their constant-distance property. You may find it worthwhile to take time to let students construct this locus in a physical way. Use two nails in a board (or pushpins in cork) as points *A* and *B*. A string roughly 6 inches longer than the distance between the pins can be looped around the pins. When drawn taut, the string not between the pins will be a constant length, allowing one to trace the locus with a pencil.

Problem 20 is the converse of earlier problems in which a situation is presented and students must find a numerical invariant. Here, the invariant is presented, and students must describe the situation that gives rise to it.



The problems in the section "Circle Intersections" in Investigation 1.18 have students make this ellipse construction using geometry software.

ASSESSMENT AND HOMEWORK IDEAS.......

Any of the problems that don't require geometry software could be used for homework: 3–8, 11, and 20.

Later activities will ask students to conduct less-directed hunts for invariants. At that time, you can assess ideas that are being introduced here. If you want to create a quiz, you can ask students to summarize ways to look for numerical invariants. (What are clues that you might look for when searching for a constant sum or product rather than a constant difference or ratio?)

WITHOUT TECHNOLOGY ....

Problems 1 and 2 and Problems 13 and 14 can be done off the computer, with each student examining several positions for the movable point on his or her circle. Make sure circle size varies across the class.

Problems 9 and 10 are important if your students will be using geometry software later in the course—they teach a way to create fixed total length—but can be skipped otherwise.

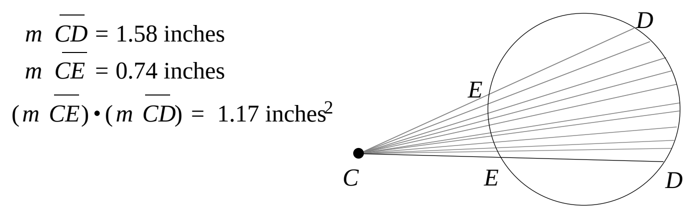
Problems 15 and 16 are important, as they introduce ideas that will recur when students study similarity. If you have only one computer, these can be done as a demonstration, with the class guessing what seems to be invariant as you or a student moves the sketch, takes measurements, and calculates the guesses.

Problems 18 and 19 can both be done off the computer, but geometry software allows students to check more cases.

MATHEMATICS CONNECTIONS ....

Problems 13–14: Given a fixed circle and a unit of measure, the "power of a point" theorem associates a real-number value with every point on the plane. A simplified statement of the theorem says that for a given point C and any line through C that intersects the circle at D and E, the product $CD \times CE$ is invariant, independent of which line through C is chosen. In the picture that follows, the values of CD and CE

will change as D moves along the circle, but their product will remain 1.17 square inches.



Does this construction produce *all* of the rectangles with this fixed area?

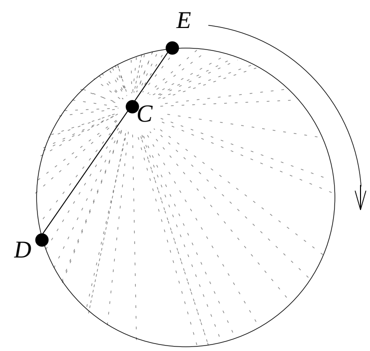
As the line through C approaches tangency to the circle, D and E approach each other. When the line *is* tangent, D and E coincide at the point P of tangency. Because $CD \times CE$ is constant, and because $CP^2 = CD \times CE$ when D and E coincide, CP is the geometric mean of CD and CE.

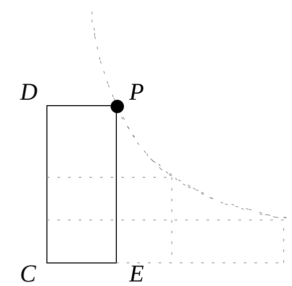
If C is *on* the circle, one of the distances to the circle is always 0, and so the product is always 0. If C is far away from the circle, both distances are large, and so C has high "power." If one interprets measurement in the conventional way—all distances are positive or zero—then the position *inside* the circle for which the product is largest is the center, which is maximally distant from the circle in any direction.

(Traditionally, the "power" of C is taken to be negative when C is inside the circle. If C is taken to be a kind of "origin," with distances measured in two opposing directions from it to the circle, then one distance is "positive" and the other "negative," with the result that their product is negative. The student problem judiciously avoids the notion of "signed distance.")

Because CE and CD can vary while their product remains constant, a rectangle built with sidelengths equal to CE and CD will have a fixed area. Moreover, if vertex C of the rectangle remains fixed, and the orientation of the rectangle does not change (it

does not rotate), then the point opposite C (that is, P in the illustration below) will follow the path of the hyperbola with equation $xy = c$, where c is the "power" of C.





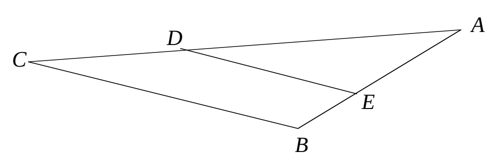
Problem 17: The ratio $\frac{DA}{CA}$ is invariant as one drags A or C. This is *not* a geometric theorem but an invariant built deliberately into the software. For some students, it may be worth pursuing the question: How can you tell whether a discovered result is a fact of geometry or a feature of the software?

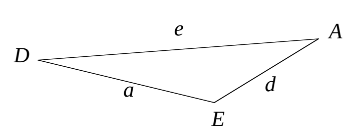
The ratios $\frac{AE}{AD}$, $\frac{DE}{AD}$, and $\frac{AE}{DE}$ are invariant as D moves. That makes their sums, differences, products, squares, and so on, invariant as D moves. The actual values of these invariants, however, depend on the shape of (which is to say, the angles in) $\triangle ABC$.

A consequence that at first looks quite arbitrary but happens to be quite important is this invariant:

$$-\frac{1}{2} \cdot \left(\frac{DE}{AE} + \frac{AE}{DE} - \frac{DA}{DE} \cdot \frac{DA}{AE} \right).$$

To make the formula easier to read, we relabel the segments as shown below:





NUMERICAL INVARIANTS *(continued)*

With this notation, our invariant becomes

$$-\frac{1}{2}\cdot\left(\frac{a}{d}+\frac{d}{a}-\frac{e}{a}\cdot\frac{e}{d}\right).$$

The value of this invariant is the cosine of $\angle DEA$, and is more usually expressed in one of the following forms, as the Law of Cosines:

$$\frac{a^2+d^2-e^2}{-2ad}=\cos\angle DEA$$

or

$$a^2+d^2=e^2-2ad\cdot\cos\angle DEA.$$

When $m\angle DEA=90^\circ$, the cosine becomes 0, leaving the Pythagorean relationship:

$$a^2+d^2=e^2.$$

In fact, one is unlikely to arrive at this formula by making arbitrary combinations of ratios and seeing what marvels they produce (nor should such arbitrariness be encouraged). But students for whom the investigation seems appropriate *can* get essentially the entire way without any knowledge of the Law of Cosines or other ideas from trigonometry.

You might plant a single seed by recalling for students the Pythagorean Theorem, which states that the quantity $a^2+d^2-e^2$ would be invariant (zero) if the triangle had a right angle at E, and wonder aloud whether this is still an invariant (though different from 0) when $\angle AED$ is *not* 90°. Experimenting shows that $a^2+d^2-e^2$ is *not* invariant. But (when it is positive) it does vary in the *same direction* as a^2, d^2, and e^2, so we might try again for an invariant by taking the ratio of $a^2+d^2-e^2$ and one of those squares. Experimentation shows that division by *any* of those squares and, for that matter, the product of any two sides (for example, ad) will give an invariant. A little algebra shows that this is exactly what should be expected: for example, after performing the division on the expression $\frac{a^2+d^2-e^2}{e^2}$, the result, $\frac{a^2}{e^2}+\frac{d^2}{e^2}-1$, is just a sum of ratios that we have already assumed (on the basis of observation) to be invariant.

NUMERICAL INVARIANTS *(continued)*

At this point, students may have generated a list of invariants such as

- $\dfrac{a^2 + d^2 - e^2}{e^2}$
- $\dfrac{a^2 + d^2 - e^2}{d^2}$
- $\dfrac{a^2 + d^2 - e^2}{a^2}$
- $\dfrac{a^2 + d^2 - e^2}{de}$
- $\dfrac{a^2 + d^2 - e^2}{ad}$
- $\dfrac{a^2 + d^2 - e^2}{ae}$.

What values do these invariants take on as the *shape* of $\triangle AED$ changes? Do any seem more "tame" than others? As it turns out, five of them vary between seemingly arbitrary limiting values (some go infinite, some have finite limits, but numbers that are not "recognizable"), while one of them,

$$\frac{a^2 + d^2 - e^2}{ad},$$

varies neatly between -2 and 2! On further investigation, this ratio is *invariant* not only when D moves but even when the triangle changes shape, as long as the angle at E stays fixed (even if the sidelengths adjacent to $\angle AED$ in $\triangle AED$ vary). These facts—especially the fact that this ratio depends solely on the angle at E—make this particular invariant more interesting than the others.

Teaching Notes

Investigation 1.16

Student Pages 108–116

SPATIAL INVARIANTS

Materials: Blackline masters of discs and polygons

Technology: Geometry software

OVERVIEW

This investigation introduces students to the hunt for spatial invariants: shape, collinearity, and concurrence.

The main ideas:

- Invariants don't have to be numbers or relationships between numbers; they can also be shapes or relationships between shapes.
- Connecting the midpoints of a quadrilateral in order produces a parallelogram.
- concurrencies of perpendicular bisectors and angle bisectors in a triangle
- concurrencies in regular and cyclic polygons
- collinearity of points equidistant from two fixed points (perpendicular bisector)

Making the required constructions for this investigation can be time-consuming if students are not already comfortable with the software.

The day before: Do you need to reserve the computer lab?

TEACHING THE INVESTIGATION

What's coming up? Problem 3 is a preview for the last section of this module, "Beyond Belief," but also contains its own content. If you plan to skip "Beyond Belief," you may want to make this a more extended investigation. If you plan to include "Beyond Belief," you can treat Problem 3 more briefly.

Day	Discussion	Homework Suggestions
Day 1	Discussion and conjecture about the set-up; Problems 1–3.	Problems 4 and 5
Days 2 & 3	Problems 6–14: You may want pairs to choose *one* of Problems 11–14 and present the results to the class. If everyone does all four problems, you will want to allow at least three days for this.	Problem 15
Day 4	Problems 16 and 17; begin working on problem 18 if there is time.	Problems 18–20, 22
Day 5	Problems 21, 23, and 24: Students work in groups and present their results to the class either at the end of class or the next day.	

Problem 5: Blackline Masters of regular polygons are provided at the end of this investigation's notes.

Problems 6 and 7 are difficult. You can ask questions about the point of concurrence. In each case, what might be special about that point? For the perpendicular bisectors, it must be equidistant from all five vertices, so it must be the center of the circumcircle. For the angle bisectors, the point of concurrence must be equidistant from all of the sides. This would be the center of the incircle.

Problem 16: Blackline Masters of discs are provided at the end of this investigation's notes. As an alternative to cutting out discs from paper, you can distribute one overhead transparency to each student and instruct the student to draw one accurate circle and its center on the transparency. (Make sure that circle size varies across the class.) On an overhead projection screen, mark two points and then have students come up one-by-one to lay their circles on the screen so that their circle's circumference touches the two points. When all of the transparencies are laid on top of each other, the centers of the circles will lie on a line.

Problems 21 and 22 suggest a minimal kind of consolidation of new words and ideas. You may want to do more with your class. The class might, for example, make an illustrated list, booklet, or bulletin board display, including the following items:

- ways to find numerical invariants
- spatial invariants students have observed
- definitions
- conjectures and theorems
- open questions

ASSESSMENT AND HOMEWORK IDEAS.......

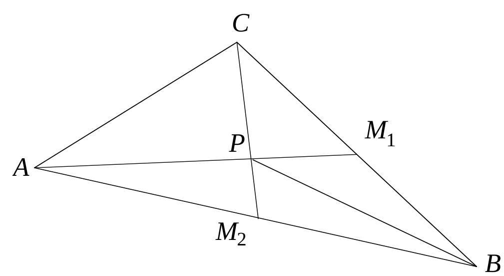
Any of the problems that don't require geometry software could be used for homework: 5, 15, 16 (as an individual rather than a group problem), and 18–24.

Later investigations will ask students to conduct less-directed hunts for invariants. At that time, you can assess ideas introduced here. If you want to create an in-class quiz, you can ask students to summarize ways to look for numerical invariants.

MATHEMATICS CONNECTIONS . . . .

The three medians of a triangle, like the perpendicular bisectors and the angle bisectors, are concurrent. The "reasoned argument" about why this is true is more difficult than for the other two. Here is one area-based argument:

Begin with a triangle, $\triangle ABC$, and two of its medians. (In the picture, M_1 is the midpoint of $\overline{CB}$, and M_2 is the midpoint of $\overline{AB}$.) The medians intersect at point P, so connect B to P.

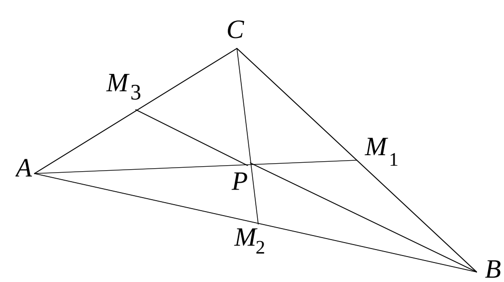


A median bisects area: The base is half the original, and the height remains constant. ("Bisect area" means to divide a region into two equal-area regions.)

Comparing areas, we know that both $\triangle CM_2B$ and $\triangle AM_1B$ have half the area of the original triangle. Removing the area of the quadrilateral PM_1BM_2 from both, we see that the areas of $\triangle CM_1P$ and $\triangle AM_2P$ are equal.

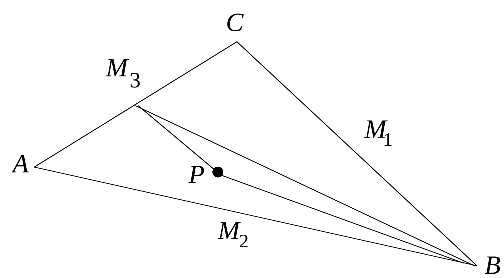
Now look at $\triangle BPC$. Its area is bisected by $\overline{PM_1}$. Likewise, the area of $\triangle APB$ is bisected by $\overline{PM_2}$. So the following four triangles have equal area: $\triangle CPM_1$, $\triangle APM_2$, $\triangle BM_1P$, and $\triangle BM_2P$. In fact, each area is $\frac{1}{6}$ the area of triangle $\triangle ABC$, since three of these equal-area pieces make up $\triangle CM_2B$, which is one half of $\triangle ABC$.

Now, connect P to M_3, the midpoint of $\overline{AC}$. We cannot just assume that $\overline{BP}$ and $\overline{PM_3}$ lie on the same line, but if they *do*, we are done, so that's what we have to show.



The area of $\triangle APC$ is $\frac{1}{3}$ the area of the whole triangle, and $\overline{PM_3}$ bisects the area of $\triangle APC$. So the areas of $\triangle CPM_3$ and $\triangle APM_3$ are each $\frac{1}{6}$ the area of $\triangle ABC$.

Now, the shape BPM_3C has exactly half $\triangle ABC$'s area because it contains three smaller triangles, each with $\frac{1}{6}$ the area of $\triangle ABC$. We know that if we draw the median $\overline{BM_3}$, it bisects the area of $\triangle ABC$.

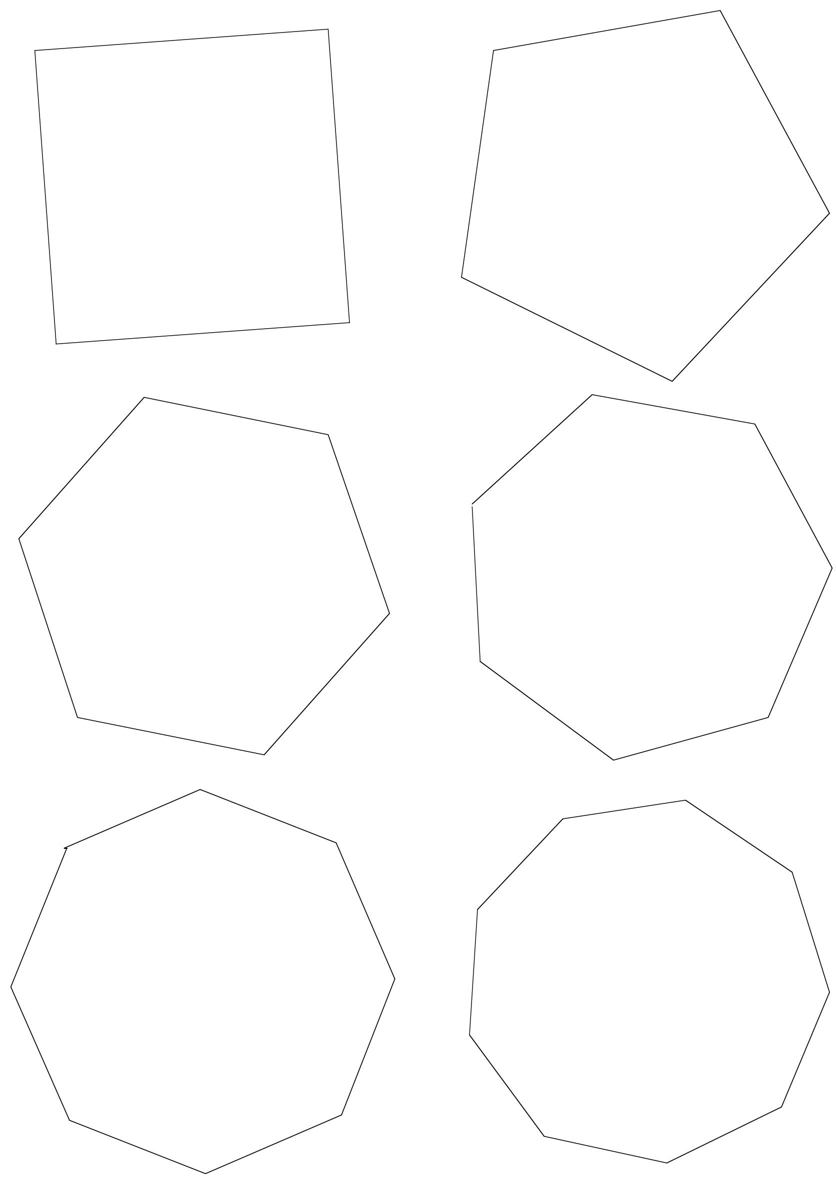


The areas of BPM_3C and $\triangle BM_3C$ are the same—half of the area of $\triangle ABC$—so P must lie on $\overline{BM_3}$, meaning that the three medians are concurrent, intersecting at P.

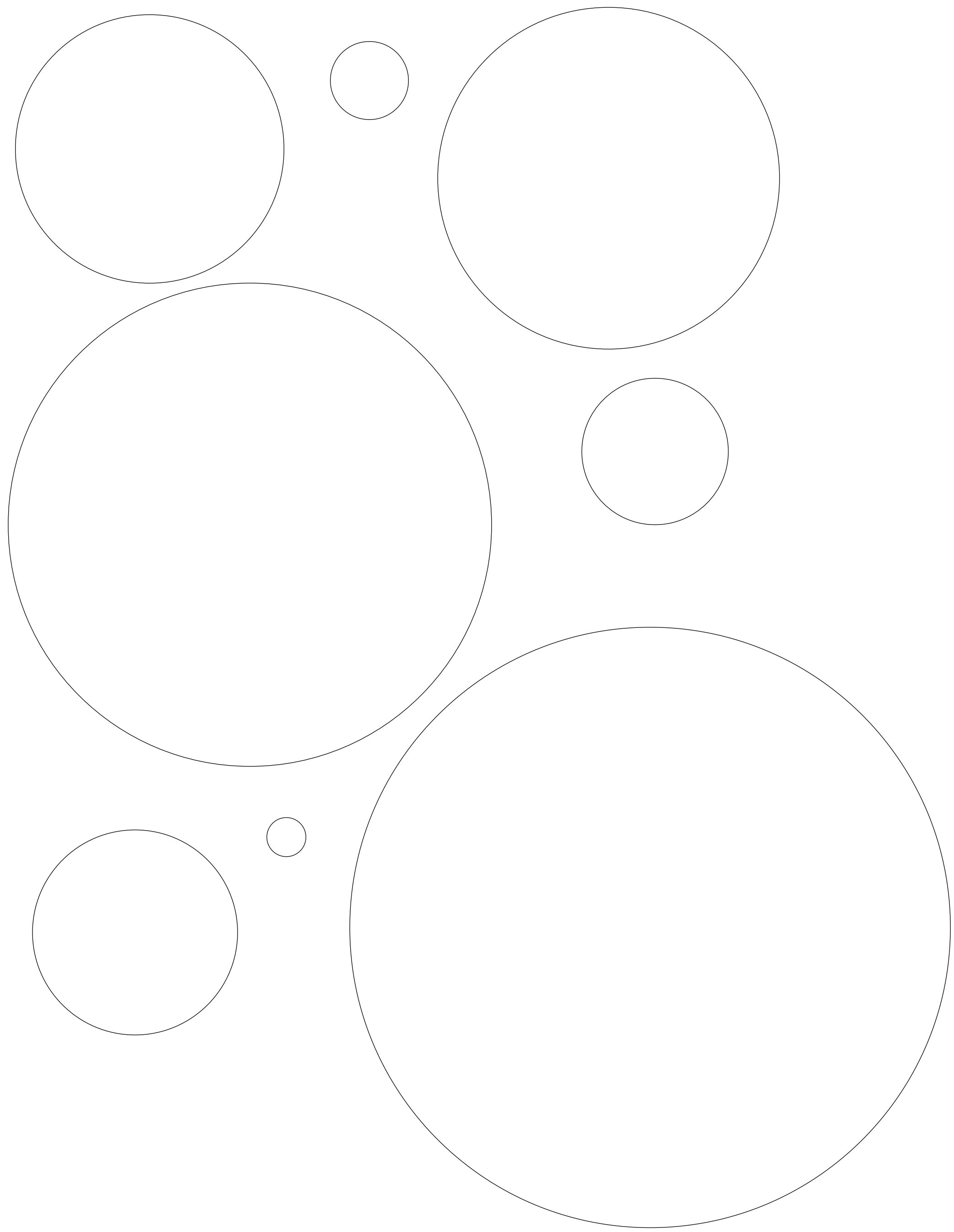
BLACKLINE MASTERS ....

Blackline masters of regular polygons and discs follow.

Regular Polygons



Discs



Parallel Lines

Technology: Geometry software

Overview

In this investigation, students learn some essential standard geometry content while they continue to hone their "invariant-hunting" skills. These results about parallel lines and angles are essential throughout geometry study. (This knowledge is prerequisite for the modules *A Perfect Match*, *The Cutting Edge*, and *A Matter of Scale*.)

The main ideas:

- definition of parallel
- angle relationships in parallel lines
- *proof* that the sum of the angles in a triangle is 180°
- looking for invariants

The day before: Do you need to reserve the computer lab?

Because the focus of the investigation is no longer on making constructions but on investigating them, it helps if students are facile at using geometry software to construct parallels, measure lengths and angles, and manipulate objects on the screen once they are constructed.

The word "congruent" is used in these Teaching Notes, but not in the Student Module, to talk about angle relationships. You should decide if you want to introduce the term to your students now or wait until they study congruence more formally. If you want to wait, you can substitute "have equal measure."

Teaching the Investigation

You can assign Problems 1–3 for homework the night before beginning the investigation in class.

Day	Discussion	Homework Suggestions
Day 1	Students should have completed Problems 1–3 for homework the night before. Discuss these problems in class so that everyone is clear on terminology. Work on Problems 4–5. Class discussion follows, especially focused on conjectures from Problem 5.	Summarize conjectures about angles and parallel lines: Which angles are congruent? Which angles sum to 180°? Use these results as a basis to prove that opposite angles of a parallelogram are congruent and that consecutive angles of a parallelogram are supplementary. (See "Notes" below.)

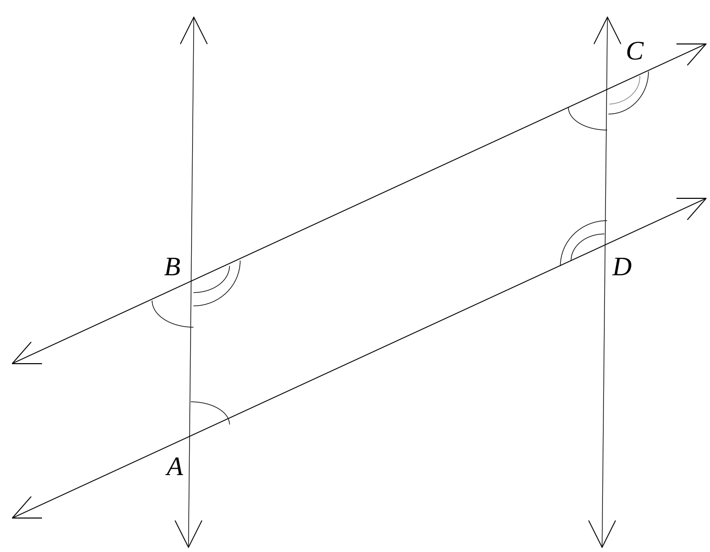
PARALLEL LINES (continued)

Day 2	Discuss proofs of the parallelogram angles. Work on Problems 6–8.	Write up conjectures from the investigations. Rewrite the proof for parallelograms, if necessary (to be turned in the next day).
Day 3	Work on Problem 10 in groups. Discuss results as a class. Work on Problem 11 in groups and then discuss it with the whole class.	Have students write up "if-then" statements about parallel lines, as many as they can come up with. (See "Notes" below for one such list.) Write up proof for Problem 11 to be turned in the next day.

Optional Project: Some students may choose to investigate Problem 9 as a project and present their results to the class.

Here are solutions for the two homework assigments suggested above:

To prove that the consecutive angles in a parallelogram are supplementary, use the figure below.



PARALLEL LINES (continued)

Because *ABCD* is a parallelogram, we know that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$. Same-side interior angles, which are exactly the consecutive angles of the parallelogram, are supplementary.

Here is half the proof that opposite angles are congruent: The interior angle at *A* is congruent to the exterior angle at *B* (alternate interior angles). The exterior angle at *B* is congruent to the interior angle at *C* (corresponding angles).

Here is a list of parallel line and angle facts that students might generate. It should include either mention of a transversal or a picture of parallel lines cut by a transversal so that it is clear which angles are being discussed.

Parallel Line Postulates

1. If two lines are parallel, then alternate interior angles are congruent.
2. If the alternate interior angles are congruent, then the lines are parallel.
3. If two lines are parallel, then the corresponding angles are congruent.
4. If the corresponding angles are congruent, then the lines are parallel.
5. If two lines are parallel, then same-side interior angles are supplementary.
6. If the same-side interior angles are supplementary, then the lines are parallel.

In the investigation, students might discover that for any two intersecting lines, vertical angles are congruent. This result is proved in the Solutions Resource for this module. Also, it is one of the first proofs students are asked to construct themselves in the module *A Perfect Match*.

The rationale for Problem 1 is that students remember terms best when the terms make linguistic sense to them. Let students discuss the names and terms, and learn where the names come from and where else (outside of geometry) they can be used.

a. Interior angles are “inside” the parallel lines. “Alternate” means they’re on opposite sides of the transversal.

b. “Alternate exterior angles” would be angles “outside” the parallel lines and on opposite sides of the transversal.

This is an important technique in mathematics: assuming a result for a proof and finding a proof for that result later, based on other assumptions.

Problem 11: In Investigation 1.15, students were asked to prove that *if* the sum of the angles in a triangle is 180°, then the sum for other polygons is $(n-2)180°$. You might point out to students that they now have proved both results, assuming facts about parallel lines and angles.

ASSESSMENT AND HOMEWORK IDEAS.......

Specific homework assignments are suggested in "Teaching the Investigation" above. For assessment, ask students to explain how they found invariant relationships. Did they find angles that sum to 180° by noticing angles that changed in opposite directions? Students' written descriptions of their investigations should provide enough information about what they've done to serve assessment purposes. If students don't write enough about the investigation, but just list results, you can ask them to add written explanations of their thinking as they work and then resubmit the problem solutions.

WITHOUT TECHNOLOGY......................

The results from this investigation are important for later investigations in this module, as well as for further study of geometry. This investigation, as written, is hard to "unplug": the use of geometry software is integral. If your students cannot do this investigation on a computer, you will need to cover the ideas about parallel lines with a lesson of your own design.

Investigations of Geometric Invariants

Overview

Investigation 1.18 is designed as open-ended activities for students to practice "hunting for invariants."

There is no need for each student to do every investigation. Instead, you could choose two or three for your class, or allow students to choose two or three on their own. Brief summaries of each activity follow to help you and your student make choices.

Because of the independent nature of these activities, separate *Teaching Notes* are provided for each investigation.

Assessment and Homework Ideas

Each of the five investigations includes problems that could be used as homework or in-class assessments. If you choose to have students present their work on one investigation as a final assessment, you might want to ensure that presentations are spread over the five investigations to reduce duplication.

Although each investigation has unique content, there are main ideas to emphasize throughout. These ideas are detailed on the first page of Investigation 1.18 in the Student Module:

- Good research technique requires thoughtful experimentation, reliable recording of results, and reflection on the results in a way that brings sense and order to them.
- Invariants (in these situations) include constant measure, sum, difference, product, ratio, shape, concurrence, and collinearity.

Midlines and Marion Walter's Theorem

Technology: Geometry software

OVERVIEW

Students use geometry software to explore midlines of triangles and then "Marion Walter's Theorem" (a theorem about connecting trisection points to opposite vertices in a triangle).

The main ideas:

- investigating a shape invariant and a numerical invariant
- similarity (informal ideas)

The day before: Do you need to reserve the computer lab?

Facility with geometry software (ability to construct midpoints, move constructions around by dragging, and take measurements) is required.

Reading about Ryan Morgan gives away results of the investigation, so it should be done after students have completed their investigations rather than between the two days.

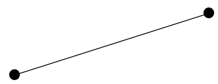
TEACHING THE INVESTIGATION

Day	Discussion	Homework Suggestions
Day 1	Students work on the "Background Check" (Problems 1 and 2).	Write up results from the background check.
Days 2 & 3	Students work on Problems 3 and 4.	Problems 5–7; read about Ryan Morgan.
Day 4	Problem 8; begin working on full write-up and presentation or on next investigation.	

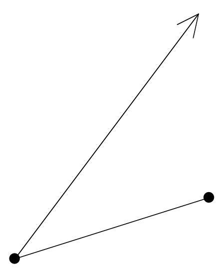
Some software provides tools to subdivide a segment into any number of pieces. If yours does not have such a tool, your students will need some way to do this.

The module *A Matter of Scale* discusses this method in detail for students.

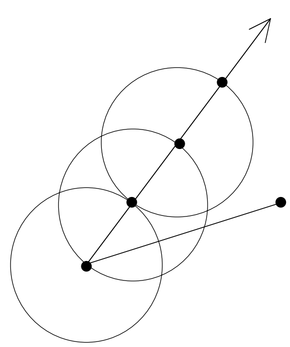
You can create a script that will trisect a segment, or you can teach your students the general method and allow them to create their own scripts. In any case, the "parallel" method is probably the easiest way to create the trisection:



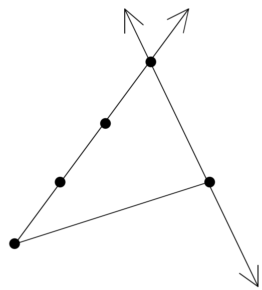
Begin with a segment.



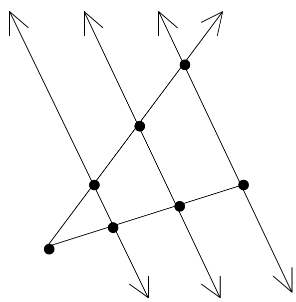
Construct a ray from one endpoint of your segment, not coinciding with the segment.



Mark off n equal lengths along that ray using circles. (In this case, we want three equal lengths, but the method works for any n.)



Connect the last mark on your ray to the other endpoint of your segment.



Construct parallels to that line, through each of the other marks on your ray. Those parallels divide your original segment into n equal pieces.

ASSESSMENT AND HOMEWORK IDEAS.......

Homework is suggested in "Teaching the Investigation".

For assessment, students should complete a final write-up summarizing their results, including how they found them. This should be done before moving on to the next investigation. It can be turned in for comments and suggested revisions. One of these write-ups will become the final presentation at the end of the "Invariants" section of this module.

A Folding Investigation

Materials: Square paper for folding (Patty paper is ideal.) See "Additional Resources" below.

OVERVIEW

Students explore a shape invariant through paper folding: If you pick a point on a square and then construct the perpendicular bisectors of the segments between that point and each of the four vertices, what shape will contain the point? Students must conduct several folding experiments, recognize some patterns, test some extreme cases, and divide up the paper into regions that behave the same way.

The main ideas:

- looking at extreme cases
- exploring a shape invariant
- perpendicular bisector is equidistant from two endpoints of a segment
- polygons

A lot of vocabulary will come up, but it can be developed as students work on the problems: perpendicular bisector, the notion of equidistant, and names for polygons. If students have done paper folding before, they will have a better idea what the folds are doing, but it is not essential.

The day before: Get materials ready.

Do you need to reserve the computer lab for day 2?

TEACHING THE INVESTIGATION

Assign Problem 9 for homework the night before you begin the investigation.

Day	Discussion	Homework Suggestions
Day 1	If students did Problem 9 for homework, they should discuss their work and compare conjectures. The discussion should lead to the conclusion that the fold is the perpendicular bisector of the segment between the marked point and the corner of the paper. Begin the investigation, working on Problems 10–13.	Finish the investigation and write up solutions.

Day 2 Work on Problem 14 if you have access to the computer lab. Otherwise, students should turn in write-ups for comments and begin work on another investigation or on final presentation.

ASSESSMENT AND HOMEWORK IDEAS.......

Homework is suggested in "Teaching the Investigation" above.

For assessment, students should complete a final write-up summarizing their results, including how they found them. This should be done before moving on to the next investigation. The write-up can be turned in for comments and suggested revisions. One of these write-ups will become the final presentation at the end of the "Invariants" section of this module.

ADDITIONAL RESOURCES ....

Patty paper is available from many publishers of mathematics manipulatives, including Key Curriculum Press. You might also ask a local butcher to sell you the paper which is used to separate hamburger patties. Waxed paper makes nice creases but it is a bit harder to handle, and you must cut it into squares. Origami paper also works but is more expensive.

Circle Intersections

Technology: Geometry software

OVERVIEW

Students use geometry software to construct circles and ellipses as the "trace" (locus) of points. Students analyze the construction to determine why these shapes emerge.

The main ideas:

- segment used as constant total length
- circle used as constant distance from a given point
- ellipse as the set of points at a constant total distance from two points

The day before: Do you need to reserve the computer lab?

Students should be familiar with geometry software, circles, ellipses (the definition can be introduced in the investigation, but it's nice if students can at least recognize and name the shape when they see it produced), and the Triangle Inequality (from Investigation 1.1).

TEACHING THE INVESTIGATION

This investigation should take about two days, and students should work straight through the problems. After the first day, you may want to hold a class discussion about the results from Problem 16. The thing that's invariant about all of the traced points is that they all lie on the same shapes. What's the name for the shape they trace out? Did anyone check special cases like C at the midpoint of your segment?

What's coming up? The fact that every point on the perpendicular bisector of a segment is equidistant from the two endpoints is proved (by students) in the module *A Perfect Match*.

After the second day, you may want to hold a class discussion on the results of Problems 17 and 18. If necessary, introduce and remind students of the name "ellipse." Analyze the construction and the definition of an ellipse to see why that shape is created.

DEFINITION

An ***ellipse*** is the set of points whose total distance from two fixed points (called the foci of the ellipse) is a constant.

Note: Even though they've encountered this before (in Investigation 1.13), you may need to remind students that naming $\overline{BC}$ as the radius of a circle does not mean it must be attached to the circle.

ASSESSMENT AND HOMEWORK IDEAS.......

Homework is suggested in "Teaching the Investigation" above.

For assessment, students should complete a final write-up summarizing their results, including how they found them. This should be done before moving on to the next investigation. It can be turned in for comments and suggested revisions. One of these write-ups will become the final presentation at the end of the "Invariants" section of the module.

Centers of Squares

Technology: Geometry software

OVERVIEW

Students investigate a surprising shape invariant: Start with any quadrilateral and build squares, facing outward, on each of its sides. If you connect the opposite centers of the squares, the segments are perpendicular and congruent, no matter what the original quadrialteral.

The main ideas:

- constructing squares and properties of squares
- perpendicular segments
- congruent segments
- investigating special cases
- informal proof

The day before: Do you need to reserve the computer lab?

Students should have geometry software experience and familiarity with properties of different quadrilaterals (squares, nonsquare rectangles, and parallelograms).

TEACHING THE INVESTIGATION

You can ask students to complete the "Background Check" (Problem 19) for homework the night before beginning the investigation.

Day	Discussion	Homework Suggestions
Day 1	Begin by discussing the "Background Check," so that everyone can proceed with the investigation. Students work on Problems 20 and 21.	Write up one of the "geometric reasons" requested in Problem 21.

Day 2	Discuss Problem 21 as a whole class. Students can share their explanations and ask questions. Work on Problems 22 and 23.	Write up two more of the "geometric reasons" requested in Problem 21. All three will be turned in the next day.

Note: You might want to provide a "Square" script so that students select the endpoints in the right order to create a square that faces outward. If your students are comfortable with the software, they may create the squares themselves or create their own script. The point of the investigation is to look at what happens after the squares are constructed, so it is best if the square-construction step does not take up a significant piece of class time.

ASSESSMENT AND HOMEWORK IDEAS.......

Homework is suggested in "Teaching the Investigation" above.

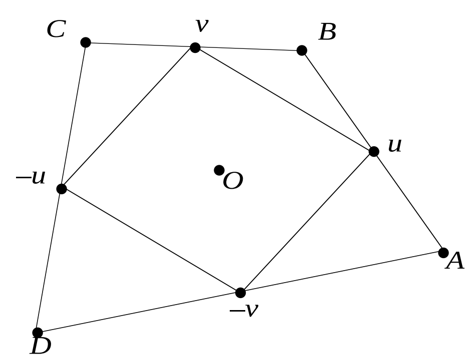
For assessment, students should complete a final write-up summarizing their results, including how they found them. This should be done before moving on to the next investigation. The write-up can be turned in for comments and suggested revisions. One of these write-ups will become the final presentation at the end of the "Invariants" section of this module.

MATHEMATICS CONNECTIONS ....

A complete solution, more complex and probably not to be found by most high school students, was created by Han Sah, a mathematician at State University of New York at Stony Brook. (See *The Cutting Edge* Student Module for a short biography of Professor Sah.)

Prof. Sah said his inspiration for this solution was the "pirate problem," *Captain Bonny's Treasure* in Investigation 1.19 of the Student Module. See the solutions for Problems 4–5 of Investigation 1.19 in the Solution Resource.

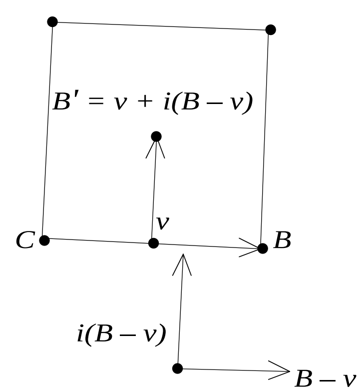
Given quadrilateral *ABCD*, the four midpoints form a parallelogram. Thus, we may assume (using complex numbers) that these midpoints are: u, v, $-u$, and $-v$. By symmetry, assume u, v are midpoints of *AB* and *BC* respectively.



The origin is marked to make things clearer.

We've taken out everything but the origin, the side $\overline{BC}$ from our original quadrilateral, and the square erected on $\overline{BC}$ to make things easier to see.

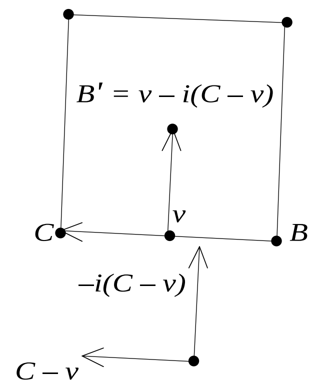
The center points of the squares you erected can each be written in two ways. Look at the point B' in the picture below.



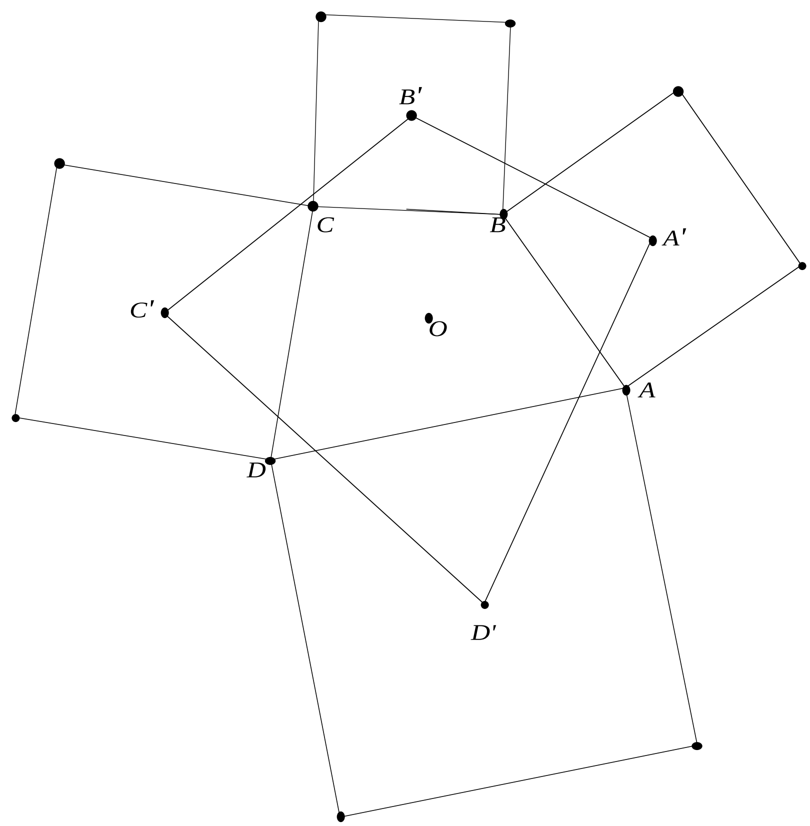
Similarly we can express each of the other three centers in two ways.

We can create the point B' by looking at the vector $B - v$ (this is a vector starting at the origin equivalent to the vector from v to B shown in the picture), rotating it by 90° (since we're in the complex plane here, that's the same as multiplying i by $B - v$), and then translating it out to start at the point v. So the point B' can be written as $v + i(B - v)$.

We can create B' in another way, though, by using the vector $C - v$:

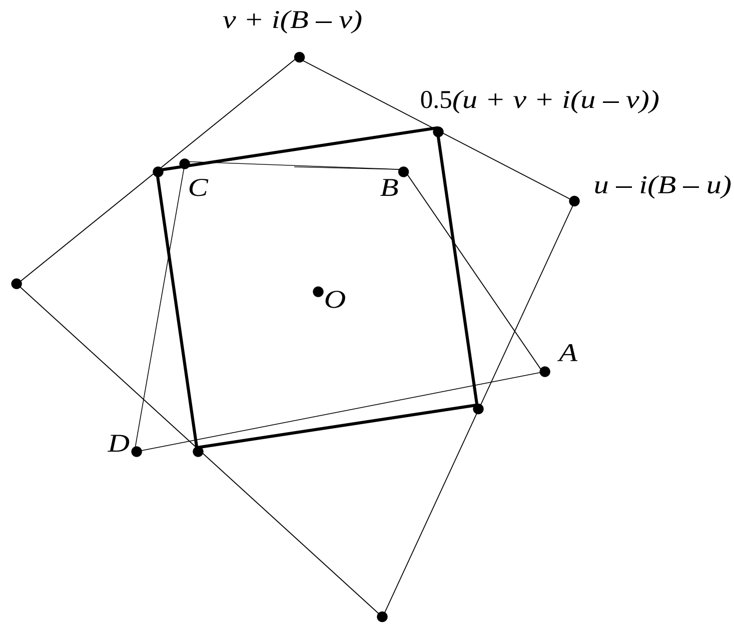


We can connect up the four centers of squares to form a new quadrilateral:



To draw the conclusion that $\overline{A'C'}$ is perpendicular and congruent to $\overline{B'D'}$, we can show

that the parallelogram formed by connecting the midpoints of the sides of $A'B'C'D'$ is in fact a square. Because we can write each vertex in two ways, we can express each midpoint in terms of just u and v. The example of the midpoint of $\overline{B'C'}$ is shown below.



Thus the four midpoints of of $A'B'C'D'$ turn out to be:

$$\frac{(u+v)}{2}+\frac{i(u-v)}{2}$$
$$\frac{(v-u)}{2}+\frac{i(v+u)}{2}$$
$$\frac{-(u+v)}{2}-\frac{i(u-v)}{2}$$
$$\frac{-(v-u)}{2}-\frac{i(v+u)}{2}.$$

The difference between the first two is $u+iv$. The difference between the second and third is $v-iu=-i(u+iv)$. These two are perpendicular and congruent, so the parallelogram formed by connecting the midpoints of the sides of $A'B'C'D'$ is a sqaure. The square's diagonals are perpendicular and congruent, so $\overline{A'C'} \perp \overline{B'D'}$ and $\overline{A'C'} \cong \overline{B'D'}$, as desired.

Constructing Invariants

Technology: Geometry software

OVERVIEW

Rather than looking for invariants in given situations, students use geometry software to create situations that contain given invariants.

The main ideas:

- independence of perimeter from area
- invariant ratios of perimeters
- invariant ratios of areas
- constant perimeter
- constant area

The day before: Do you need to reserve the computer lab?

Students should by now be quite comfortable with geometry software and constructions. They will need to use circles, squares, and rectangles in their constructions, and they will need to measure length and area and make calculations.

TEACHING THE INVESTIGATION

Day	Discussion	Homework Suggestions
Day 1	Students should work through as many of the problems as possible, taking notes on the constructions and making sketches where necessary to remember what was done.	Write up detailed solutions, including pictures, for one or two of the problems.

Day 2	Check homework by making the described constructions on the computer to be sure they work. Revise if necessary. Continue working on problems.	Write up revised directions to be turned in the next day. Write directions for one or two more of the problems.
Day 3	Turn in the first set of directions, check the second set, and make revisions if necessary. Work on more problems. By the end of day 3 or 4, all students should have worked through several problems.	

Last Day: Students who solve Problems 27–30 explain their solutions to the class in informal presentations. If there is a problem no one in class solved, you can provide step-by-step directions for students to make a construction that contains the invariant.

Notes: Because there are many different ways of solving Problem 25, you might challenge some students to invent ways that they think are particularly "elegant" (simple constructions), or "unusual" (approaches they think few others will find), or that have ratios other than $\frac{1}{2}$ or $\frac{1}{4}$.

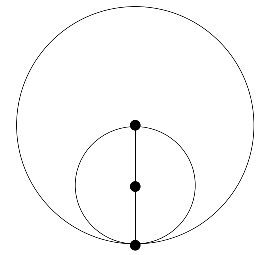
ASSESSMENT AND HOMEWORK IDEAS.......

Homework is suggested in "Teaching the Investigation" above. Throughout work on this investigation, students should also be preparing one of the five investigations in Investigation 1.18 for presentation to the class.

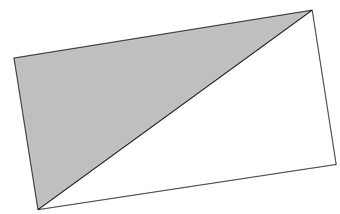
WITHOUT TECHNOLOGY ....

Problems 24–26 can be done without the computer. If students know enough about area and perimeter of shapes, they can see from a hand drawing that the invariant is independent of the size of the drawing or how it is moved around:

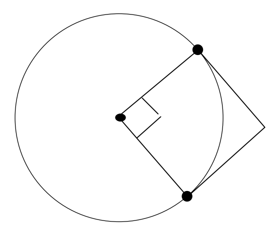
INVESTIGATIONS OF GEOMETRIC INVARIANTS
(continued)



$C_2 = 2\pi r_2 = 2C_1$



Area of the shaded triangle is exactly half the area of the rectangle.



The area of circle is πr^2. The area of square is r^2. The constant ratio is π.

Teaching Notes

Investigation **1.19**

Student Pages 138–142

GUESS-AND-CHECK

Materials: Graph paper and copies of the grid in the Student Module for "Finding Your Homework."

Technology: Calculators, spreadsheet software, or software like Function Machines or Logo are needed for the "jeans and shirts" problem. Geometry software enhances, but is not necessary for, the "Captain Bonny's Treasure" section.

The day before: If you are using computers for any of the problems, do you need to reserve the computer lab and/or install the software?

You may want students to "act out" the story. Set up the landmarks and have two students pace out the directions.

OVERVIEW

This investigation contains four quite different sections: "Spending the Most," "Finding Your Homework," "Captain Anne Bonny's Treasure," and "More Problems."

- To use a guess-AND-CHECK strategy on the jeans and shirts problem requires so little calculation that other techniques are unnecessary and really more work.
- The hidden homework game *requires* guessing—there's no other way of gaining information—but it focuses on how well one makes use of the feedback.
- Anne Bonny's hidden pirate treasure provides a good example of how a guess gives *insight* into a problem. By experimentally seeing what effect one's choice of starting places has on the location where one later digs for the treasure, it is easy to solve the problem.

Coordinates are introduced in "Finding Your Homework." They are not a prerequisite, but if students have used them before, the investigation will move more quickly.

TEACHING THE INVESTIGATION

Day	Discussion	Homework Suggestions
Day 1	Students work on Problem 1, the jeans and shirts problem, using a calculator, spreadsheet, or other software to make the calculations and check their guesses. Depending on students' familiarity with the technology and their experience solving these types of problems, this may take most of a class or just a few minutes. If your students solve it quickly, move on to Day 2.	"Finding Your Homework" or Problems 6 and 7
Day 2	As a class, read through the setup and play of the "Finding Your Homework" game. You may want to play a sample game with a student as a demonstration to make sure that everyone understands how to play. Students play several games in pairs, taking turns "hiding." If there is time, move on to Problem 3.	Write up results of the game (Problem 2).

GUESS-AND-CHECK *(continued)*

Day 3	Read the story of "Captain Bonny's Treasure" as a class. Work in groups on Problems 4 and 5.	Explain your results. Why is guess-AND-CHECK a good way to approach these problems?

The notes for Problems 4–5 in the Solutions Resource include a proof using coordinates. Don't expect your students to come up with this reasoning at this point unless they have had significant experience with coordinates. Problems 6 and 7 in the "More Problems" section may *seem* to say that fancy mathematical methods—techniques from algebra, for example—are not important. In fact, for *specific* problems, guess-AND-CHECK methods *can* be just as good. The real power of algebraic methods (or ideas from number theory or other branches of mathematics) lies in their power of generalization—their ability to solve a whole class of problems with one technique and with equal ease. With easy numbers, Problems like 6 and 7 require no special techniques. But special techniques *guarantee* answers and make it no more work to get those answers for hard numbers than for easy ones.

Imagine this: I found 30 pieces of homework, one for each student in the class, and I want to put each piece of homework in the correct box. But I've forgotten my glasses, and the clerk will help me only with directions like "up" or "down and left," etc. I want the minimum total number of steps for the entire procedure.

For the "Finding Your Homework" game, you must make clear to students the purpose, which is to develop a strategy rather than just play a game. Students are looking for a rule of thumb that guarantees success for a one-shot problem, with success measured by the minimum number of steps. Different kinds of searches optimize for different things. The "divide by two" strategy optimizes for total number of guesses but not for finding the homework fastest. If there were some reason that the student wanted to find it more quickly, another strategy (which may take more than five guesses for guaranteed finding but optimizes for fewer) would be more appropriate. Students, especially those interested in programming or computer science, might be interested in a project on different search strategies.

Also for "Finding Your Homework," the rules don't state that the homework must be hidden at a crossing on the grid. The "hider" could just as easily choose a location like $(\frac{1}{2}, 5\frac{2}{3})$ (or even $(\sqrt{2}, 2\pi)$). Of course, the problem is not solvable (there is no "minimum number of guesses") if such locations are used. Several students will almost certainly come up with this idea on their own, trying to trick their partners. This provides an excellent opportunity for discussion about how this case is different

from lattice points. Don't give it away; students feel very clever coming up with this on their own.

For Problem 3, consider asking these questions:

- Is the y-coordinate less than 2? If so, the homework is "south" of a line drawn across at $y = 2$.
- Is the product of the coordinates a number greater than 16? If so, then the homework is toward the outer part of quadrant I or quadrant III.
- Is $x^2 + y^2 < 25$? If so, the homework is within a circle of radius 5 centered at $(0, 0)$.

The methods that students use to solve Problem 7 are often quite similar to what they might write algebraically. For example, one student might describe picturing "two pencils, and the extra \$0.89" and work from there. For some classes, it may be valuable and interesting to compare algebraic and such "nonalgebraic" methods. Students may see that the "nonalgebraic" methods, if they produce a correct result, involve many elements of algebraic thinking, omitting only the formal symbols.

ASSESSMENT AND HOMEWORK IDEAS.......

- The section on "More Problems" can be used for homework anytime.
- "Finding Your Homework" can be used for homework: Ask students to play with a friend or relative, write up the results, and bring in their write-ups the next day. Then hold a class discussion about their ideas for a best strategy.
- Problem 8 can be used for assessment. You may want to give students a few problems from which to choose rather than a whole algebra text. They should also explain in writing what the guess-AND-CHECK strategy is, why it's useful on these problems, and how you alter a guess based on the outcome of the CHECK.

USING TECHNOLOGY ....

Problem 1: These are some Logo programming ideas for the jeans and shirts problem.

Programming Idea 1:

Here the amount of money spent on `jeans`, the amount spent on `shirts`, the total `cost`, and the amount `remaining` are all separate functions.

GUESS-AND-CHECK (continued)


```
to Jeans :jnum
 output :jnum * 29.95
end

to Shirts :snum
 output :snum * 15.99
end

to Cost :j :s
 output (jeans :j) + (shirts :s)
end

to Remaining :jeans :shirts
 output 250 – cost :jeans :shirts
end
```

Type `print Jeans 7` or `print Shirts 3` to find out how much 7 jeans or 3 shirts would cost. Or type `print Cost 7 3` to get the total cost of 7 jeans and 3 shirts. Type `print Remaining 7 3` to see the amount you have leftover from your original $250 (or how far over you've gone) after a purchase of 7 jeans and 3 shirts.

Programming Idea 2:

The Logo procedure can be translated directly from the algebraic expression for the problem: $R(j, s) = 250 - (29.95j + 15.99s)$.

```
to Try :jnum :snum
 output 250 – (:jnum * 29.95  +  :snum * 15.99)
end
```

Type `print Try 5 2` to see what the value of $R(5, 2)$ is, and then try other values to find the lowest.

Programming Idea 3:

This is an embellishment that will appeal to some students' sense of elegance (and desire to program) but goes beyond what is necessary for solving the problem. Using this approach, students can specify a number of jeans and let Logo tell them how many shirts can be bought with the remaining money, as well as how much money is left unspent. There are many ways of doing this. The program suggested below makes use of all the procedures from **Idea 1** and adds some new ones.

```
to AfterJeans :jn
 output 250 – (jeans :jn)
end
```

GUESS-AND-CHECK (continued)


```
to HowManyShirts :moneyleft
 output integer (:moneyleft / 15.99)
end

to ShirtsWJeans :jn
 output HowManyShirts  AfterJeans :jn
end

to FigureItOut :jn
 print (sentence [If I buy] :jn [jeans,])
 print (sentence [then I can buy] ShirtsWJeans :jn [shirts,])
 print (se [and have] (word "$ remaining :jn ShirtsWJeans :jn)
 [leftover.])
end
```

AfterJeans tells how much is leftover after buying the jn pairs of jeans. HowManyShirts tells how many shirts can be bought (an integer!) with a given amount of (remaining) money. ShirtsWJeans puts these together to calculate how many shirts can be bought along with a given number of jeans. Typing print ShirtsWJeans 5, for example, should result in the computer typing 6 as the number of shirts that can be bought. FigureItOut 0 does its own printing (no need to type "print figureitout 0"), and the result is:

```
If I buy 0 jeans,
then I can buy 15 shirts,
and have $10.15 leftover.
```

Reasoning by Continuity

Technology: Geometry software would be useful, but is not necessary for the investigation. See "Using Technology".

The day before: If you choose to use geometry software, do you need to reserve the computer lab?

What's coming up? Reasoning by continuity is a major theme of the *Connected Geometry* module *Optimization*.

Overview

Students examine the difference between continuous and discrete change, and learn to use some big ideas of continuity in solving problems.

The main ideas:

- the Mean Value Theorem (not named, but the ideas are used)
- noticing when a situation does or does not change continuously

"The Box Problem" discusses volume. To explore the problem as presented in the Student Module, students need at least an informal notion that volume measures "how much stuff it will hold." If students know more—good algebraic skills and the formula for volume of a box—they can explore the problem more deeply.

Informal notions of area and volume are used in the "Ham Sandwich Problem" as well, but again no formulas are needed.

Teaching the Investigation

Day	Discussion	Homework Suggestions
Day 1	Students work on Problems 1–3 in groups. A class discussion about the results (Problems 4 and 5) should follow. Key issues about the differences between continuous change and discrete change (Problems 1 and 2) should come out of the discussion. Students should then work on Problem 6.	Write up answers to Problems 4–6.
Day 2	A fun way to introduce the "Box Problem" is to give your students an incentive to create the largest possible box. Give individuals or groups several cards and 15–20 minutes to create the largest box they can, using the method of cutting squares out of the corners and folding up the sides. At the end of that time, fill their box (whichever one they think is largest) with candy.	Students should explore and answer the following problem: A cross-country runner runs six miles in 30 minutes. On average, that's a mile every 5 minutes, but surely the runner was not running at an absolutely constant speed!

		Perhaps, in fact, there was *never* a 5-minute period during which the runner ran exactly one mile. What do you think? Justify your answer.
Day 3	Groups work on the “Ham Sandwich” problem in two dimensions. Afterward, hold a class discussion about the continuously-moving line and the informal notion of the Mean Value Theorem.	Problems 17 and 18
Day 4	Discuss homework from previous two days, clarifying any questions. Do computer experiments (see “Using Technology”).	

Some students may consider Problem 2 to be a continuity problem like the previous one. Some others may see that the population may jump suddenly if a family of 8 moves into town. The distinction is important, and is raised in the “Write and Reflect” questions.

The question posed in Problem 3 anticipates the next investigation by introducing issues about definition (see, especially the comments to that problem in the Solution Reference).

Problems 7–10 (The Box Problem): Students’ first impulse is sometimes a bit scattered —cutting squares, building boxes, and measuring volumes with little plan or reflection. Here is an opportunity to help them develop good *scientific* habits along with their mathematical reasoning: encourage them to keep a record of the measurements; present the data so that they can be viewed at a glance (for example, in a table or graph); if using a table, *organize* the data so that some relevant *order* is visible (for example, placing entries in order by the size of the cutout or by size of the volume).

REASONING BY CONTINUITY *(continued)*

While the students eat their candy, they can create a table of the cutouts they made and the volumes of the boxes produced. Students can then answer Problems 8 and 9. (Each student should make a new box, possibly using the same cutout size as before but more likely changing it.)

Volume could be measured in candies, with sand, or by a calculation if students know the formula.

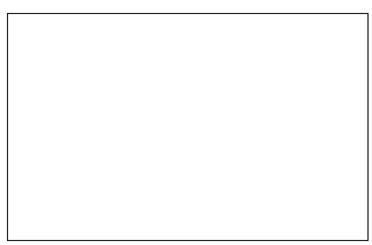
In recent years, The "Box Problem" has become widely used in curricula. One of the reasons is that it can be approached at so many levels—there's an easy entry for students who do not yet know elementary algebra, yet the mathematics that can be applied extends well into calculus. Because this *is* such a widely-used problem, a comment on *our* particular purpose and focus seems warranted.

The techniques of calculus allow one to arrive at a precise symbolic solution. It is also possible to generate a numerical solution to very high precision with simple algebraic equations and a graphing calculator. But the problem specifically asks students—even those for whom this would be quite a reasonable task—*not* to use formulas or numbers. Why?

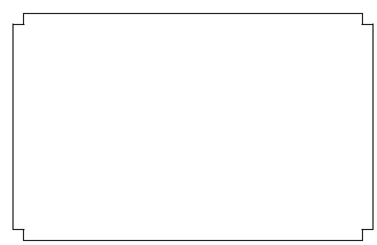
Our goal is certainly *not* to bypass or downplay algebraic and analytic techniques. But, *for now*, the goal is to get students to understand some implications of continuous change, and for that reason we want them to think about the problem using only its continuity.

For some classes, it may well be a very valuable second step to apply algebra to the problem, but the first step should not be cut short.

We would like students to picture several note cards like the four that follow:

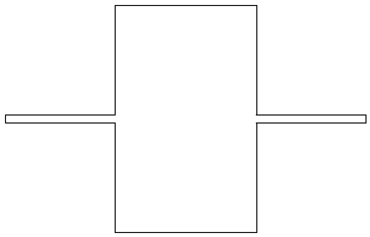


No cutouts, no volume

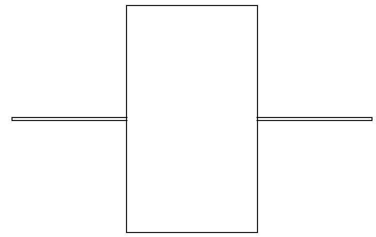


Tiny cutouts, tiny volume

REASONING BY CONTINUITY (continued)



Huge cutouts, tiny volume



Completely cut away

For the intermediate cutouts, the volume gets bigger for awhile as the cutouts get bigger, and then the volume starts to shrink. Where is a good guess for the maximum volume? Once you have a starting guess, you can check numbers close by with a calculator to try to get even bigger.

With some reasoning by continuity, we can be sure that there is a point at which a cutout will produce the desired box of maximum volume. Students can make quite good approximations for the dimensions of this box either by further experimentation or by using some geometry software. Some software allows the students to see how the box's volume and shape change as they change the size of the cutouts. Depending on the algebra background of the class, students may set up the function, graph it, and use the graph to approximate the maximum volume.

$V = x(4x^2 - 26x + 40)$

People can learn to perform experiments like these in their head—mathematicians call them "thought experiments"—but very few people have had the opportunity to develop these visualization/analysis skills. One way is to gain experience doing the experiments physically. Much of our emphasis on geometry software is intended for the same purpose—not to generate the *answers* so much as to let students see with their eyes the kinds of continuously changing systems that they will later learn to recreate, modify, and extend in their minds.

Problem 12 asks students how they could know if they've been successful at bisecting irregular shapes. The fact is that they can't know. There's no area measuring device to check them and no handy way to compute the areas of the two halves. Students need the theory to tell them that it's even possible, and then the best they can do is approximate.

For the homework problem suggested on pages 121–122, here are some ideas to discuss with students:

> Imagine a graph of the distance the runner traveled over the previous five-minute period, beginning at the 5-minute mark and ending at the 30-minute mark. The values, if they are not all identical, must at some point dip below 1 mile during the five minutes and at some other point rise above that value. Between those two places, there must be *some* point at which the graph crosses the 1-mile value. At that point, the runner has just completed a mile in five minutes!

Assessment and Homework Ideas.......

Homework is suggested in "Teaching the Investigation." For assessment, the homework problem suggested for Day 2 is a good way to see how students think about continuous variation. Also, Problem 18, while difficult to complete, allows you to see students' thinking: Do they use planes instead of lines? Can they reproduce the two-dimensional argument in the three-dimensional case for one object? For two objects?

Using Technology ....

A geometry software sketch of "Captain Bonny's Treasure" shows strikingly that the location of the treasure does not depend on the location of the gallows. This can be done as a demonstration after students work through some hand drawings on their own and come up with the conjecture, or students can investigate the problem on the computer. The sketch should be easy for students to construct by now:

It's probably helpful to hide construction lines like the perpendiculars and the circles after you use them.

- Create three points; label one "gallows" and the other two "trees."
- Draw segments from the gallows to each of the two trees.
- Construct circles about the trees, using the segment to that tree as a radius.
- Construct perpendiculars to each of the segments.
- Find the intersection of the perpendiculars with the circles—one with a left turn and the other with a right turn.
- Connect those two intersection points with a segment.

REASONING BY CONTINUITY *(continued)*

- The midpoint of that segment is where the treasure is buried. Drag around the gallows. What happens to the location of the treasure?

The box problem is also nicely modeled with geometry software. If students realize that the volume is simply three lengths on their sketch multiplied together, they can have the software calculate it for them dynamically (and even graph volume *vs*. size of cutout). A favorite sketch created by one of the authors shows a dynamically-changing rectangle (with cutouts), box, and graph all on the same page. Look at the Internet at

http://www.edc.org/LTT/GAMT/box.html

to see a sample of this sketch.

Definitions and Systems

Materials:
- **models of spheres (globe, Lénárt Sphere™)**
- **string**
- **wires (or string) and beads (or clay)**

Overview

This is a fun hands-on investigation that includes a fanciful story. The main idea is that there are geometric systems other than the familiar Euclidean geometry usually studied in high school, and that what is true in one system may not be true in another. This idea is extremely important, even for students who will not continue the studies of axiomatics or alternative systems.

Students should have enough experience with Euclidean goemetry that this investigation has meaning for them.

Students should be encouraged to use string, held at the two points and pulled taut, for both Problems 2 and 3.

Teaching the Investigation

Day	Discussion	Homework Suggestions
Day 1	Read the introductory page. In groups, come up with answers to Problems 1–3. (Working in pairs or groups will be important especially on Problem 3 because it is difficult to work with the sphere models and string individually.) End class with a discussion about the meaning of "straight" on the surface of a sphere.	
Day 2	Students work on Problems 4–7 and Problem 9.	Problems 8–10 (with a careful write-up of the work done in class on Problem 9).
Day 3	Work on Problem 11, read the story as a class, and work in groups on Problems 13–15.	Optional: Students work on Problem 15 as a project.
Day 4	Pass out the beads and wires (or similar materials) and write the four "rules" on the board. Students work in pairs or small groups to create the described object. Groups draw their creations on the board and discuss similarities. They should then work on Problems 17–19.	Read the "Perspective" and answer Problems 20 and 21.

DEFINITIONS AND SYSTEMS
(continued)

Here are some questions for the discussion prior to Problem 1: What is "straightness"? Is it the path that light travels? What about on the surface of the earth? Perhaps it is a vertical projection onto the surface from the path that light travels. You could also consider a person or vehicle with right and left feet (or wheels) traveling the same distance. Or is it a "straight" path the shortest path between two points?

If two roads are straight like rays of light, then they won't stay on the earth. We need to give a meaning to "straightness on a sphere." A helpful tool for students would be a globe or Lénárt Sphere (see "Additional Resources"). Great circles (including the equator, the longitude lines, and arbitrary great circles) are shortest-distance paths, and are the "straightest" lines on a sphere. Latitude circles are helpful in establishing coordinates on the globe, but do not follow shortest-path routes, and are thus not "straight" in the usual meaning of the word. Bringing a globe to class and asking students to find "straight" lines on it may generate controversy, as some students may at first choose both the longitude and latitude circles as the spherical "straight lines." You might also ask them to find two actual longitude circles that start very close to each other, "one block apart," on the equator and see whether the distance between them remains constant. They will soon see that the two lines are forced to meet on the North and South Poles, so that the distance between them is not constant.

Problem 5: Problem 1 has already hinted that any two "straight lines," as they've been defined on a sphere, cannot remain the same distance apart and must eventually intersect. Thus, parallel lines do not exist on a sphere. Later in this section, students will see that, while parallel lines (in the mathematical sense) cannot exist on a sphere, perpendiculars can. This leads to a rather startling conclusion, and gives room for a lively debate about the existence of squares on a sphere. One side of the debate argues correctly that four equal-length line segments can be constructed, and that it is possible even to build in a right angle. That conforms to one definition of a square. But squares, at least in plane geometry, are also a very special kind of parallelogram, and parallelograms can *not* exist on a sphere (because parallels can not exist on a sphere)! How to resolve this? One must make a decision about which properties to use in *defining* squares for the purpose of doing geometry on a sphere.

Students who use the globe as their model of a sphere may be confused by the geographical use of "parallel" as in "the 49th parallel." This terminology makes a different compromise from the one in our discussion. The senses of non-intersecting and everywhere-equidistant are preserved. But the "parallels" cannot both be "straight lines."

If one decides to define a square as an equilateral four-sided polygon, then squares exist on a sphere, but their properties are not the same as on the plane. For example, they have no parallel sides and no right angles. If one decides, equally reasonably, to define a square as a kind of parallelogram, then a square cannot exist on a sphere.

Problem 6 asks students to investigate the angles in triangles on a sphere. Students

Goniometers are angle measurers. Built out of two pieces hinged together with something like a protractor at the hinge, goniometers can be used more easily than protractors in many situations.

will likely need help with the idea of measuring these angles. Goniometers or other such devices are probably preferable to protractors, but even with a protractor, students can get an idea of the angle measure with some help. It's probably best that students work in pairs on this problem (and on all the problems using the spherical models) so that they can negotiate the difficult points with another student rather than struggling alone.

ASSESSMENT AND HOMEWORK IDEAS.......

- Some homework ideas are suggested in "Teaching the Investigation" above.
- Problem 11 is a good assessment: Do students examine the geometry of a sphere or overgeneralize from the plane? What kinds of questions do they decide to investigate? They should look for existence of shapes or properties they've learned. (For example: in a plane the set of points equidistant from a given point is a circle; what is that set on a sphere? If there *are* circles, what about inscribed angles?)
- Problem 19 is also a good assessment problem, this one for the finite geometry.

ADDITIONAL RESOURCES ....

Key Curriculum Press sells the Lénárt Sphere^{TM}. It is a clear sphere that can be drawn on with grease pencils. It comes with an angle-measuring tool and hemisphere overhead projection slides.

MATHEMATICS CONNECTIONS ....

Problem 11: There are many possible directions the explorations might take. Students who are interested in *definitions* may find that the problems of redefining objects on this new surface do not end with lines (great circles) and squares (see above). One must even reconsider what a "point" is. In the plane (and, for that matter, in the 3-dimensional space of our experience), two intersecting lines define a single point; and two distinct points completely determine a (unique) line. That is such a "natural" idea that we'd want it to be true in other geometries as well as on the plane. But it appears *not* to be the case on a sphere. Two distinct lines (great circles) intersect at *two* antipodal points. And, while "most" pairs of points on a sphere determine a single great circle, there are many pairs of points (*any* pair of antipodal points) through which an infinite number of lines pass. To salvage the orderliness that had felt so natural in

Euclidean geometry, spherical geometry defines its (spherical) points differently: a pair of antipodal Euclidean points is considered *one* (spherical) point. So, any pair of lines intersects in a single point, and any two distinct points determine a single line. Students who are interested in *phenomena that remain invariant under changes from plane to sphere* may explore some of the concurrences and collinearities they have seen in earlier problems.

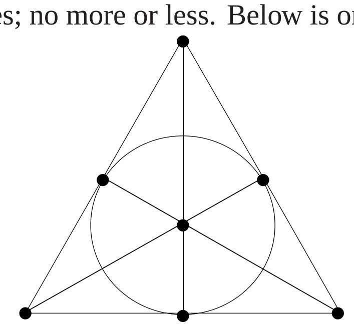
The Solutions Resource presents an approach that does not depend on radian measure.

In Problem 15, the simplest way of expressing the relationship requires measuring the angles in radians, which will be unfamiliar to most students. The familiar 360° equals 2π radians, so a triangle with three right angles contains $270^\circ = \frac{3\pi}{2}$ radians. Its area is $\frac{1}{8}$ the area of the entire sphere. The surface area of a sphere is $4\pi r^2$, so a sphere with radius 1 will have surface area 4π. Thus, the triangle with three right angles is $\frac{\pi}{2}$.

Angle Sum	Triangle Area
$\frac{3\pi}{2}$	$\frac{\pi}{2}$
$\frac{4\pi}{2}$	π
$\frac{5\pi}{4}$	$\frac{\pi}{4}$

If this pattern holds, then the area is the angle sum minus π.

In Problems 16–19, students are asked to explore a new system, the seven-point geometry. Students may come up with different models of this geometry, but they all have to be isomorphic. That is, they must have the *same structure* even if they have different appearances. Two geometric models **A** and **B** are *isomorphic* if there exists a one-to-one correspondence between their points P (on **A**) and P' (on **B**), and a one-to-one correspondence between the lines such that P lies on l (of **A**) if and only if P' lies on l' (of **B**). Students do not need to know the exact definition of isomorphism, but they should check to see that the different models they invent have the *same structure*, despite possibly different appearances. So, for example, all the models should have seven beads and seven lines; no more or less. Below is one possible model.



Non-Euclidean Geometries

Overview

This is an optional investigation designed to give students a sense of some mathematical history, both in the building up of theorems from postulates and in the history of non-Euclidean geometry. Be sure students complete Investigation 1.21 before attempting Investigation 1.22.

Teaching the Investigation

Students should read the history (individually or aloud in a group) and then discuss answers to Problems 1 and 2.

If students want to pursue the project in Problem 3, there are several places to look: The *Connected Geometry* module *A Matter of Scale: Pathways to Similarity and Trigonometry* has a short section on self-similarity. James Gleick's book *Chaos* provides a wonderful history of the field and the personalities involved without too much sophisticated mathematics. Mandelbrot's *The Fractal Geometry of Nature* is the classic text, but only parts are accessible to high school students. Ivars Peterson's *The Mathematical Tourist* is a great book for the general reader. Clifford Pickover's book *Computers, Pattern, Chaos, and Beauty* has some very accessible chapters and some more difficult chapters. Also, the National Council of Teachers of Mathematics sells a book called *Fractals for the Classroom* by Peitgen, Jurgens, and Saupe.

Additional Resources

Gleick, James. *Chaos*. Penguin USA, 1988.

Mandelbrot, Benoit. *The Fractal Geometry of Nature*. New York: W. H. Freeman & Co., 1988.

Peitgen, Jurgens, and Saupe. *Fractals for the Classroom*. Reston, VA: National Council of Teachers of Mathematics, 1982.

Peterson, Ivars. *The Mathematical Tourist*. New York: W. H. Freeman & Co., 1989.

Pickover, Clifford. *Computers, Pattern, Chaos, and Beauty*. St. Martin's Press, 1991.

VISUALIZATION EXERCISES

OVERVIEW

With Investigation 1.23, students begin developing the habit of mind of proving. The problem students investigate is a rather surprising invariant: When you connect the midpoints of any quadrilateral in order, the shape seems to always be a parallelogram. Lots of data will certainly convince students that you do, in fact, always get a parallelogram. But how can you be *sure* there's not *some* type of quadrilateral for which it doesn't work? And, anyway, why in the world would such a thing happen? The proof helps students answer both questions.

It may seem that we are doing things out of order: we use the "Midline Theorem" (the fact that a segment connecting two midpoints of sides of a triangle is parallel to the third side and half as long) to prove the result about quadrilaterals, but the students haven't yet proved the Midline Theorem! This is, however, how mathematicians work; they say, "If I knew this was true, I could prove my result." Then they go back and try to prove these lemmas. The Midline Theorem is proved, using a congruent triangle argument, in the module *The Cutting Edge*.

This is also how life works. Problems don't ask you what you know when they present themselves, no matter what field they come from. You may have the tools necessary, or you may need to learn something along the way.

Investigation 1.23 contains three "warm-up" visualization problems designed to get students thinking about the invariant they will investigate in this part. Before beginning the investigation, have students do Problems 1–3 for homework.

TEACHING THE INVESTIGATION

Ask students to think of examples where convincing conjectures don't necessarily lead to a proof. Here's one: "All numbers are odd." One can carefully pick many examples of numbers that are odd. Or "All odd numbers are prime. Look! Three is, 5 is, 7 is, 11 is, 13 is . . . That's five examples already!" If students object that the counterexamples are too easy to find because there are so many of them, try this: "Every number has an inverse, a number which it can be multiplied by to give the product 1." The inverse of $\frac{1}{2}$ is 2, the inverse of $\sqrt{3}$ is $\frac{\sqrt{3}}{3}$, the inverse of 10 is 0.1, and so on. It would seem as though there are no exceptions at all. In fact, there is only one exception: zero.

MIDPOINTS IN QUADRILATERALS

Technology: Geometry software

OVERVIEW

Students experiment with connecting midpoints in quadrilaterals in various ways and look for invariants in these situations. This experiment will get students thinking about invariants *within* a figure and invariants *between* two figures. In doing this investigation, students are exercising all their skills of looking for invariants.

Facility with using geometry software to search for invariants is needed. Students will need to create figures, drag points, construct midpoints, and measure (lengths, areas, and ratios). Vocabulary: midpoint, quadrilateral

The day before: Do you need to reserve the computer lab?

Ask students to read the first page and do Problem 1 for homework.

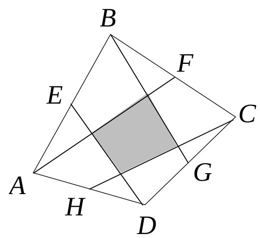
What's coming up? Students will spend the rest of this section of the module answering the question posed in Problem 6. Depending on your goals and the nature of the class, you may encourage students to follow their own leads with your mentorship, provide some partial structuring with your own supplementary problems, or proceed directly to the remaining investigations in this section.

TEACHING THE INVESTIGATION

Day	Discussion	Homework Suggestions
Day 1	Students should have completed Problem 1 for homework. In class, students move on to Problem 2 on the computer, taking notes and drawing pictures. As a class, read the conjecture and discuss parallelograms.	Write up Problem 3 and properties of parallelograms useful for determining them (for easy reference on Day 2).
Day 2	Problems 4 and 5; discussion of Problem 6	

Some of the constructions from Problem 1 are good examples of the fact that "many true cases don't make a theorem." For example, if students construct the figure below, measure the ratio of the area of the outside quadrilateral (light) to the inside quadrilateral (dark), and then change the shape of the outer quadrilateral by dragging one of its vertices around, the ratio *may* appear to remain 5. There are, in fact, an infinite number of places at which each vertex can be placed, where the ratio *is* exactly 5. But there are also places where the ratio is greater than 5. Students who want to

investigate phenomena like this can trace a vertex as they wiggle it, watching the ratio and moving the vertex back into the “5” region whenever it strays out.



The area ratio is *often* 5:1.

Problem 2: Some students may find the open-endedness of this problem daunting. For them you might provide some structuring like the following supplementary problem.

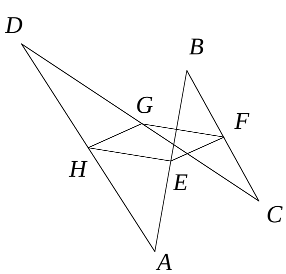
1. Restrict yourself to a special case of the outer quadrilateral and see what invariants you can find in that restricted situation. For example, what happens to the inner shape when the outer shape is:

a. a rectangle?

b. a parallelogram?

c. a rhombus?

ABCD is no longer a conventional quadrilateral, yet connecting the midpoints in order still creates a parallelogram.



If students are using geometry software for Problems 4–6, they examine extreme cases, like the one shown here. Experimental evidence from continuous measurements of sides or angles, and from the extreme cases should be *convincing*. They may, at this point, begin to look for reasons *why* this works.

ASSESSMENT AND HOMEWORK IDEAS. . . . . . .

As always with geometry software activities, writing up the investigation—with pictures, conjectures, and so on—is an ideal homework assignment. This is also an assessment relating to students’ “invariant hunting” habits. In the investigation in Problem 2, they should show their habits of looking for things that don’t change or that change in the same or opposite directions.

This presentation may come before, after, or instead of Investigation 1.25, which provides some models of presentations, as well the opportunity to critique presentations that were not done by friends.

Another assessment option is for students to prepare and deliver a presentation to the class that will convince them that *EFGH* is *always* a parallelogram (no matter what quadrilateral *ABCD* looks like). When students are listening to the presentations, remind them to keep asking themselves if the presentation is convincing. (They might have their own reasons to be convinced, but do the reasons *given in the presentation* convince them?) Students should be encouraged to help the presenters: “If they do not convince you, make sure they know why. Tell them what you don’t believe and ask probing questions. Make them explain themselves. If they do convince you, lend a hand; tell others in the class why you are convinced.”

WITHOUT TECHNOLOGY . . . .

This investigation can be carried out with pencil and paper, but it is more tedious and less convincing. A better alternative, if you don’t have access to a computer lab, would be a single computer demonstration.

Teaching Notes

Investigation **1.25**

Student Pages 166–169

What Do You Find Convincing?

Technology: Geometry software would be helpful but is not necessary.

Overview

This is an optional investigation for students who need more practice reviewing arguments or students who have difficulty proceeding on their own. Some teachers who've decided to use this investigation have had their own students make presentations before beginning it. Others have chosen to use this investigation as a warm-up to making such presentations in class.

Students read and critique two arguments that connecting the midpoints of a quadrilateral in order creates a parallelogram, no matter what quadrilateral you start with. Both arguments are data-driven, but one is clearly more convincing than the other.

Teaching the Investigation

Alternate beginning: Ask one student to take the part of Raphe, reading that explanation, and another student to take the part of Liza.

You might ask students to read the dialogues and answer Problems 1 and 2 for homework before beginning the investigation.

Day	Discussion	Homework Suggestions
Day 1	If your students are comfortable reading on their own, you can assign the first two problems (reading the explanations and critiquing them) for homework. Begin class with a discussion about students' reactions to the dialogues. Also talk about the "For Discussion" issue. If there is time, students can begin Problems 3 and 4.	Investigation 1.26, Problems 1 and 2
Day 2	Continue working on Problems 3 and 4.	

The discussion topic is really *not* about mathematics. The problem is how to interpret evidence, and it occurs daily in all sorts of different contexts. In this case, the evidence in question is a number provided by a computer tool. If you trust the tool, evidence

like this could remove any reason to even make the conjecture. But perhaps there is something inaccurate in the tool. That expression of distrust in the software might reasonably be based on the fact that your eye tells you that *something* is going on, and that the first piece of software seemed to confirm your impression.

ASSESSMENT AND HOMEWORK IDEAS.......

In Investigation 1.27, students will start including reasoned arguments, and not just convincing data, in their presentations.

This investigation will help students create better presentations themselves by getting them to think about what is convincing data and what is not. Assessment will come when they do their own presentations, either at the end of this investigation or later in this section of the module.

USING TECHNOLOGY ....

Problems 3 and 4 are best done with geometry software. You can skip them and move on to Investigations 1.26 and 1.27 if you do not have the software; the meat of the investigation is in the dialogs.

Teaching Notes

Investigation **1.26**

Student Page 170

Finding Other Invariants

Materials: Hand construction tools, including rulers

Overview

This investigation, like the previous one, is optional. It is designed for students who need more practice reviewing arguments or for students who have difficulty proceeding on their own.

Several problems are given as scaffolding for the Midline Theorem, which students will need to progress through the rest of this section of the module. Students may have conjectured the Midline Theorem already, in which case this investigation can be skipped or shortened.

This result is used in the next investigation to explain the mysterious parallelogram conjecture.

The main ideas:

- the Midline Theorem
- quadrilaterals as two triangles sharing one side (the diagonal)
- perimeter

The following vocabulary is used in the problems: *perimeter*, *conjecture*, *quadrilateral*.

Teaching the Investigation

Students can do Problems 1 and 2 for homework the night before. Then discuss the results in class and move on to Problems 3–5, using the conjecture from Problem 2. If students have not completed Problems 1 and 2, they may begin by making constructions (either by hand or with geometry software) and making the conjecture. Working in groups (especially if using hand construction tools) would make this piece of the activity proceed more quickly.

ASSESSMENT AND HOMEWORK IDEAS.......

Any of the problems can be used for homework. Problem 2 is the key to this investigation. A final in-class assessment could be to produce a written answer to it in their own words.

USING TECHNOLOGY ....

Geometry software could be helpful in the first few problems and in making conjectures, but by later problems students should be reasoning directly from the conjecture about the length of the midline and answering questions without making any constructions at all.

Teaching Notes

Investigation **1.27**

Student Pages 171–173

MAKING THE RIGHT CONNECTIONS

Technology: Geometry software, if you plan to do the "Take It Further"

OVERVIEW

Students analyze a presentation that uses the midline conjecture to explain the fact that when you connect consecutive midpoints in a quadrilateral, you get a parallelogram.

The main ideas:

- assuming a lemma to prove something else
- explaining and proving vs. patterns in data

The day before: Students read the dialog and answer Problem 1.

Investigation 1.24 and familiarity with the midline conjecture are prerequisites.

TEACHING THE INVESTIGATION

If students have not read it, ask three students to read the dialog aloud for the class. The class can then continue the discussion started by the dialog.

If students have read the dialog and answered Problem 1 for homework, begin with a discussion about this problem: Who took Dale's side? Who took Barbara's side? Why? Also discuss Problem 2. Homework: Students answer Problems 3 and 4 in writing. Students who are skilled with geometry software may want to try the "Take It Further" on the second day.

It is common for students to misunderstand the mathematical need for proof as some kind of obsessive behavior, lack of trust, or unreasonable doubt in the face of overwhelming evidence. It may help to have a discussion in connection with this investigation. Here are some ideas that may be useful in such a discussion:

For example, all numbers except zero have a multiplicative inverse.

Because the number of quadrilaterals is infinite, no amount of checking can show that *all* of them will produce a parallelogram, but few people feel any real need to check every case before becoming convinced that the phenomenon is real. Of course, situations do exist in which things are true for "nearly all" but not absolutely all cases.

The mathematician's need for proof arises not from a culture of doubt and distrust, but from two other concerns: First, if an observation is to be used as the basis for further work, then *no* uncertainty at all can be left in it, because such uncertainty would then propagate into the future work and all its potential descendents as well, resulting, soon enough, in a set of "facts" that would be completely untrustable. Second, a purpose of mathematics is to bring order to diverse observations, to build systems out of ideas, to seek insight. Thus, even when very little rests on the actual truth of the assertion—as

little rested on the truth of Fermat's Last Theorem—good mathematical thinkers seek reasons for things.

Problems 3–4: Anisha's conjecture was based on her knowledge that the opposite sides of a parallelogram are congruent. Had she instead known only that the opposite sides of a parallelogram are parallel, she could work in a similar manner: $\overline{EH}$ and $\overline{FG}$ are both parallel to $\overline{BD}$, so they are parallel to each other.

If students have performed the experiment in Problem 4, they will almost certainly come up with the conjecture that the outer figure closes only when *ABCD* is a parallelogram. Because this conclusion is so likely, the Solutions Resource gives a suggestion for an experiment that may lead some students to an even more general conjecture rather than providing the answer for this problem.

Assessment and Homework Ideas.......

Most of the problems can be used for homework, allowing flexibility in how your class moves through this investigation. Problems 1 and 2, however, deserve some significant class discussion time.

Students should start work on this final assessment now and have it completed by the time they finish the next two investigations.

A final assessment for this section of the module could be:

- a complete write-up of the "midlines in quadrilaterals" experiment, including proof in students' own words and explanation of the midline conjecture,
- a class presentation (perhaps in groups rather than as individuals) of the experiment, or
- a 2–3 day investigation of one of the other ways to connect midpoints in quadrilaterals (from Investigation 1.24, Problems 1 and 2) and a class presentation.

Teaching Notes

Investigation **1.28**

Student Pages 174–175

Can You Say More?

Overview

This short investigation is designed to start answering the question, "Why prove things?" Students see that proof can convince you that a result will *always* hold, can give you insight into why it holds, and can help you ask and answer interesting further questions.

Students should complete Investigations 1.24 and 1.27 before starting this one.

Teaching the Investigation

This investigation can be done for homework or in one day of class. Students work on the two problems and read the section on "Student Mathematicians." You should end with a discussion about proof. In this case, what did the proof do for you that the experiment and data did not? First, it let you know that there's no special quadrilateral for which you won't get a parallelogram. Second, you have an idea for *why* it's true that you always get a parallelogram. Third, you can answer questions like Problems 1 and 2 by thinking about diagonals rather than by experimentation.

Teaching Notes

Investigation **1.29**

Student Pages 176–177

THE MIDLINE THEOREM

Technology: Geometry software is helpful.

The day before: Do you need to reserve the computer lab?

OVERVIEW

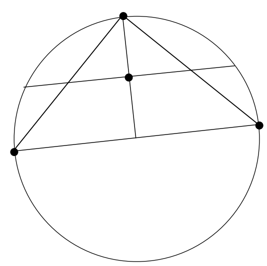
Students examine the Midline Theorem more closely. Reasoning by continuity is used to explain (but not prove) it.

Students should complete Investigations 1.24, 1.27, and 1.26 before attempting this investigation.

TEACHING THE INVESTIGATION

Read the argument in class and discuss Problem 1. Students can work in groups on Problems 2–4. At the end of the day or the next day, ask groups to present their answers to one of the three questions.

If you believe the triangle midline conjecture, then it's easy to see that the circle can't behave the same way:



The height of the triangle is the same as that of the semicircle. The base of the triangle is the same as the circle's diameter. Halfway up the height of the triangle, the segment inside the *triangle* is half as long as the circle's diameter, so the segment inside the *circle* must be longer than that.

ASSESSMENT AND HOMEWORK IDEAS

As a final assessment for this investigation, ask individuals or groups to present their solutions to Problems 2–4.

Solutions

Investigation
1.1

Student Pages 1–10

Problem Solving in Geometry

Firefighters, civil engineers, and others who know a lot about public utilities may be able to tell you more.

Problems 1–2 *(Student page 1)* The geometric properties of a shape can serve practical purposes in design or construction. Manhole covers are round—cylinders with very small heights in most cases. Roundness also allows this heavy object to be rolled out of the way when it is removed. A circular base requires no attention to orientation in order to fit it into the hole, and poses no risk of falling through the mouth of the hole if not oriented properly. (Utility covers that are not circular almost always cover shallower holes into which people reach but do not descend.)

Most, but not all, fire hydrant nuts are regular pentagons. We have also seen hexagons and squares, but they are far less common. For public safety, it is important that fire hydrants be easily used by those authorized to do so and hard to use by others. Any irregular shape for the nut would restrict access to only those with the appropriate wrench, but many irregular shapes would require the wrench to be oriented only in very particular ways. By choosing a regular pentagon, firefighters have five angles at which to orient the wrench.

Cans often have ridges at the seams and may have corrugated sides for strength, so they are not *strictly* cylindrical, but for now, don't worry about "decorative" details. The major features are what's important.

For Discussion *(Student page 1)* A soup or drink can is an obvious example of a cylinder. A coin and a piece of uncooked spaghetti are two less-obvious examples. Both have the essential features of a cylinder (circular base and the height), but are often overlooked as examples of "cylinder" because their dimensions are so extreme—extremely low height compared to the base diameter, or extremely low diameter compared to height, respectively. Some other examples: coins, manhole covers, hoses, straws, wires, blood vessels, and toothpaste extruded from a tube.

Problem 3 *(Student page 1)* You can find the exact number of handshakes for your class by using the formula $\frac{n(n-1)}{2}$. The reasoning below shows how you can figure this rule out for yourself.

One way is to look at the pattern for small numbers of people.

Class Size	3	4	5	6
Handshakes	3	6	10	

Strategy 1: See how one additional class member changes the number of handshakes.

Starting with small numbers and building a table leads to a very sophisticated strategy. In trying to figure out how many handshakes for 6 people, picture that sixth person entering the class after all the others had finished their 10 shakes. How many more handshakes are now required? Only 5: the new person must shake hands with all the old people. More generally, the nth person to enter shakes only $(n-1)$ other people's hands. So if there are n people in your class, the number of handshakes is $1+2+3+\cdots+(n-1)$.

Strategy 2: Count one's *own* handshakes, multiply, and then correct for duplications.

Another way to think about this is to imagine doing the experiment in your class. How many times would *you*, *personally*, have to shake hands to get everyone in the class? And how many people in the whole class would be doing this experiment? Multiply those two numbers (the $n-1$ handshakes that any one person made by the n people who were all doing the same thing). This counts each handshake twice, so the result is $\frac{n(n-1)}{2}$.

"-gon" means *angle*. Its relatives in English include *knee*, *genuflect*, *genuine*, and *orthogonal*, where "ortho-" means "right." "Trigon" doesn't appear by itself, but is part of "trigon o metry," which means "measuring triangles": an apt description of trigonometry.

Problem 4 *(Student page 1)* This is essentially a pictorial variation of the handshake problem. Either of the strategies suggested above would work here.

Think of the vertices of the polygon as the people in your class. Then all of the connecting segments represent all the handshakes—the problem solved earlier. The expression $\frac{n(n-1)}{2}$ computes the number of "handshakes." But this problem asks only for the number of *diagonals*, so we eliminate the outside segments (because they are not "diagonals"), and we have the answer. The expression $\frac{n(n-1)}{2}-n$ then "eliminates the outside segments" by subtracting n.

This result applies to all polygons, not just regular polygons, using the same argument.

THEOREM ***Diagonals of Polygons***

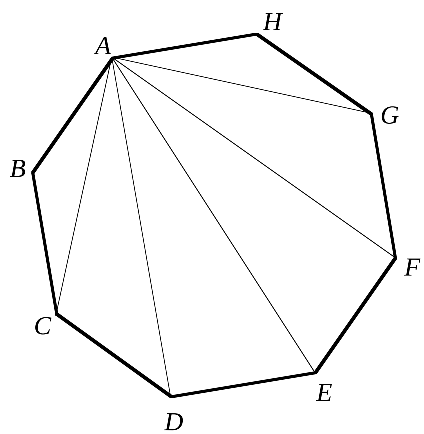
An n-gon has $\frac{n(n-1)}{2}-n$ diagonals.

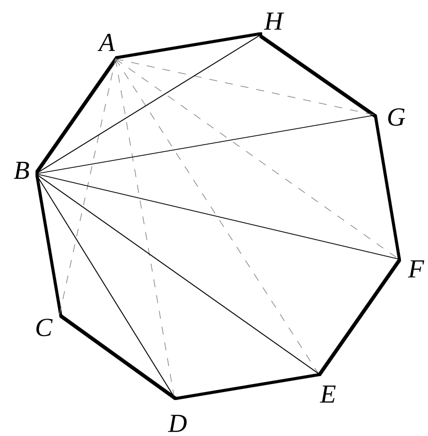
Name of Polygon	square	pentagon	hexagon	heptagon	octagon	n-gon
Number of Sides	4	5	6	7	8	n
Number of Diagonals	2	5	9	14	20	$\frac{n(n-1)}{2}-n$

There are many ways of writing rules for the number of diagonals. Here is one way: "For an n-sided figure, add up $n-3$ of these":

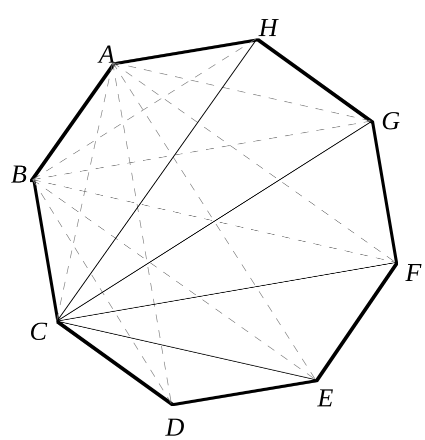
$$2+3+4+5+6+\cdots.$$

A similar rule could be derived by experimenting with a polygon in a systematic way and noting the pattern. For the octagon shown below, five diagonals can be drawn from vertex A, and five more diagonals can be drawn from vertex B.





After that, each new vertex produces one fewer diagonal.



Adding them all up, one gets

$$5 + 5 + 4 + 3 + 2 + 1.$$

Combining the last entry with the first, one gets

$$6 + 5 + 4 + 3 + 2.$$

The same rule can be written in this fancy way:

THEOREM *Diagonals of Polygons, version 2*

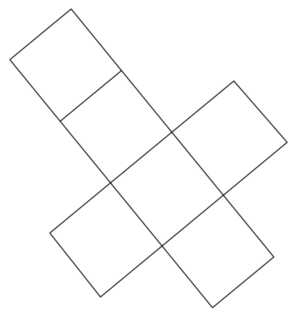
If n is the number of sides of the polygon, then the number of diagonals is

$$2 + 3 + 4 + \cdots + (n - 2).$$

For sums like this, mathematicians use a capital sigma (the Greek letter equivalent to S) for "sum." Using sigma notation, this sum can be written as

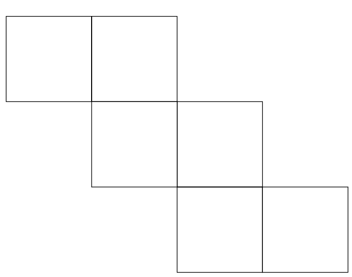
$$\sum_{k=2}^{n-2} k.$$

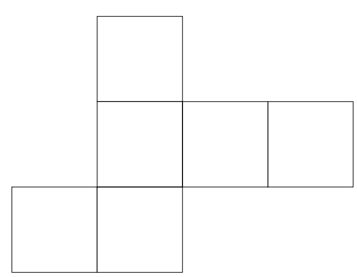
Problem 5 *(Student page 2)* The partially unfolded cube shown above this problem will flatten out to the following shape:



A cube has six square faces, so a net of a cube must contain six squares. No more than four of these squares can be in a single straight strip. (Experiment to see why.) Starting from a strip of four squares, the last two squares can be attached as top and the bottom to any square in the strip. Thus, only the last two figures in the Student Module are possible nets for a cube.

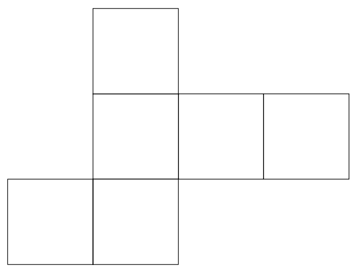
Problem 6 *(Student page 3)* Here are some of the ten possibilities.

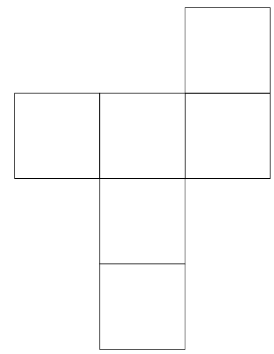




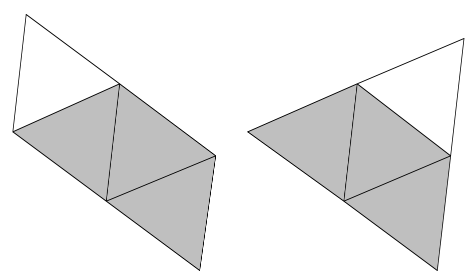
In the Connected Geometry module *A Perfect Match*, this is a congruence test for two-dimensional figures.

Because nets for cubes are two-dimensional, a possible check for "sameness" would be to cut them out and see if you can rotate, flip, or slide them so that they exactly coincide. If you can, they are the same. If you can't, they are different. For example, the following two nets are the same:

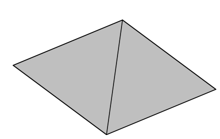




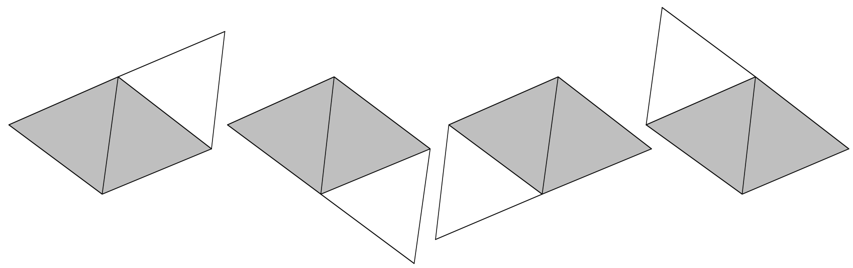
Problem 7 *(Student page 3)* There are only two distinct nets for a regular tetrahedron.



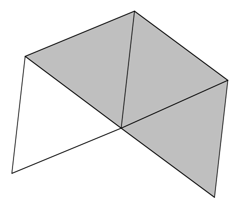
Here's one way to think about it: The faces of the tetrahedron are four equilateral triangles. After two have been glued together, we have the following:



All possible places to glue on a third triangle are really "the same" in the sense that they can be superimposed exactly on one another. (They are *congruent*.)



We pick any one of these figures, and try all the places to glue a fourth. Two of them work: the ones we have shown as solutions. The only other one (below) fails.



Problems 8–10 *(Student pages 3–4)* From the experiment, you will find that three rods will make a triangle if the sum of any two of their lengths is greater than the third length. We can state this result algebraically: If a, b, and c are three positive numbers such that $a + b > c$, then it is possible to form a triangle with sidelengths a, b, and c.

A closely-related statement, which we call the *converse* of the above statement because it reverses the "if" and "then" parts, is a well-known theorem:

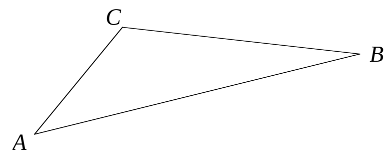
THEOREM *Triangle Inequality*

In a triangle, the sum of the lengths of any two sides of a triangle is greater than the length of the third side.

In the dice-rolling experiment, what is the probability of getting a "good" roll—a roll whose three lengths will form a triangle?

Why is it true? On any triangle with $\overline{AB}$ as a side, there are two paths from A to B—one along $\overline{AB}$, and one detouring through C. The *shortest* path from one vertex

to another is the straight path, along the side that connects them. Any other path (the sum of the two other sides) must be longer.



This explanation does not depend on the sides being integer lengths. The same is true for any sidelengths, even 0.01 or $\sqrt{2}$.

Problem 11 *(Student page 5)* The position of the vertex varies, but if the drawings and measurements could be perfectly accurate, the measure of the angle would always be 90°.

Problems 12–13 *(Student page 5)* Correctly-drawn figures appear to be rectangles. How is this connected with the observation of Problem 11?

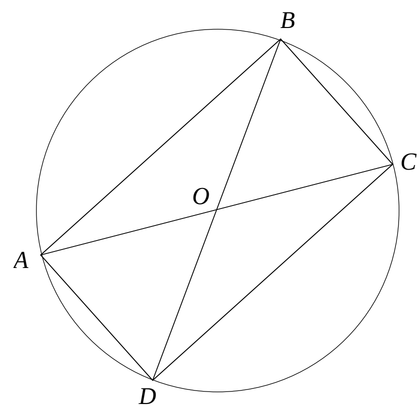
Suppose that the figures you see in Problem 12 really *are* rectangles. Either diagonal of the rectangle divides the circle into two semicircles. Two of the "corners" of the rectangle are angles inscribed in those semicircles. So you have observations that support each other:

- Angles inscribed in a semicircle are right angles.
- Connecting the four endpoints of two diameters creates a figure with four right angles.

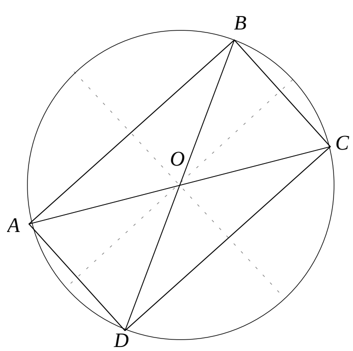
But how could we know that these figures really *are* rectangles? Here is one way to *reason* that they are.

Symmetry!

All four half-diameters are radii of a circle, so they have the same length.



$$AO = BO = CO = DO$$



***m∠ABC* = *m∠BAD* = *m∠CDA* = *m∠DCB* (*m∠ABC* is read "the measure of angle *ABC*.")**

The symmetries of a circle (and a bit of reasoning) show that folding along either of the dotted lines (at the left) folds not only the half-diagonals onto each other, but also all the other parts of *ABCD* onto each other. Folding along one dotted diameter matches $\angle ABC$ with $\angle BAD$ and $\angle DCB$ with $\angle CDA$. Folding along the other dotted diameter shows that the other pairs of adjacent angles have equal measures, so *all four angles have the same measure*.

The remaining step requires a fact that has not yet been established: The sum of the (interior) angles in any quadrilateral is 360°. Using that fact, one can now show that if the four angles are equal, each must contain 90°, so *ABCD* is a rectangle.

Problem 14 *(Student page 5)* An angle inscribed in a quarter circle has a measure of 135°; an angle inscribed in a semicircle has a measure of 90°; an angle inscribed in a three-quarter circle has a measure of 45°. This leads to one kind of general rule that might be expressed algebraically as $i = 180(1 - f)$, where i stands for the measure of the inscribed angle and f is the fraction of the circle in which the angle was inscribed.

But there are other general rules. What you find depends on how you interpret the idea of "an angle inscribed in part of a circle." The problem does not say clearly what mathematicians mean by that idea, so your experiments may lead you to other good rules.

Vocabulary: central angles, inscribed angles

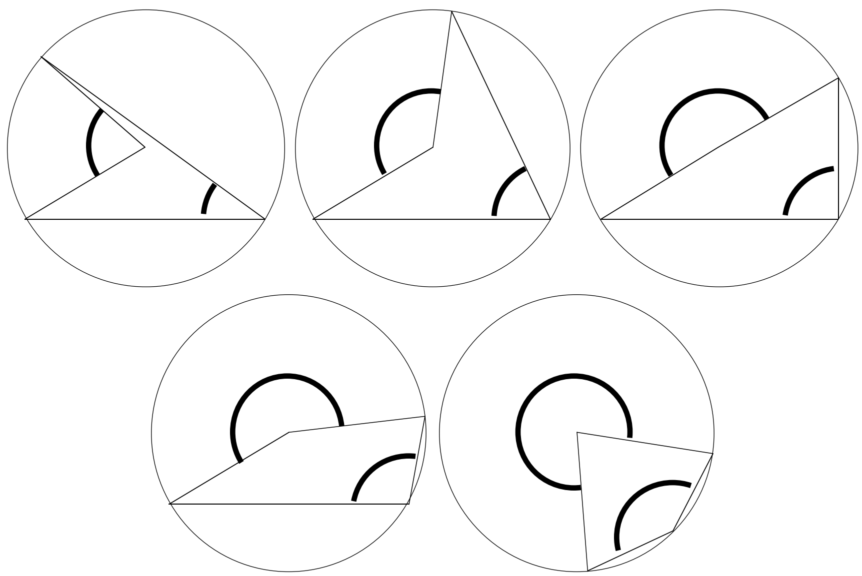
The proven relationship ("theorem") found in most geometry texts depends on what mathematicians call the "central angle" and the "inscribed angle." These are the

marked angles in the pictures below. Looking at the angles this way, the theorem states:

THEOREM ***Inscribed Angles***

The measure of an inscribed angle is always half the measure of the corresponding central angle.

The pictures illustrate that relationship.



Problem 15 *(Student page 5)* The number of positions for the vertex is infinite; they can not all be checked. A hundred cases *does* justify great confidence—enough to make a conjecture—but it doesn't guarantee what will happen in the infinitely many other cases.

Statements like "*every* angle inscribed in a semicircle is a right angle" are very strong and call for a logical argument. That is what *proof* is for.

Problems 16–18 *(Student page 6)* People often find it very difficult to visualize these cross-sectional shapes.

a. A square: Cut parallel to any face.

b. An equilateral triangle: Slicing off a corner of the cube will produce a triangular cross section. If that slice is perpendicular to a diagonal of the cube (not a diagonal of a face!) the triangular cross section will be equilateral.

c. A rectangle that is not a square: Slice parallel to an edge, but not parallel to a face.

d. A triangle that is not equilateral: Slice off a corner of the cube, but do not slice perpendicular to a diagonal of the cube.

e. Irregular (but not regular) pentagons are possible. (See “hexagon” below.)

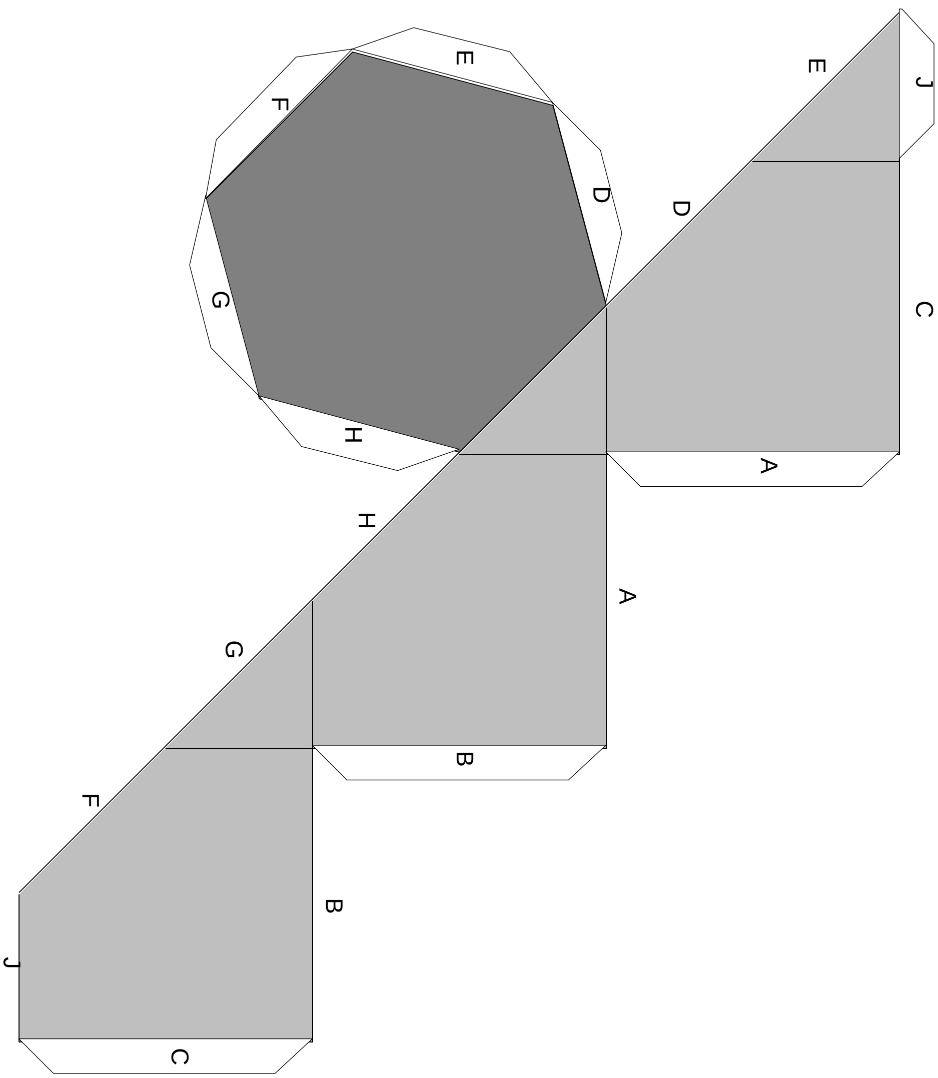
f. A hexagon: The plane that bisects and is perpendicular to a diagonal of the cube (not of a face) will cut the cube with a regular hexagonal cross section. Adjustments to the angle of that plane can create irregular hexagons or, by cutting one fewer edge, create a pentagonal (but not regular) cross section. The nets on the following pages show how a cube can have hexagonal and pentagonal cross sections.

g. An octagon: To make an octagon, the plane would have to intersect eight edges of the cube. This is not possible.

h. A parallelogram that is not a rectangle: not possible

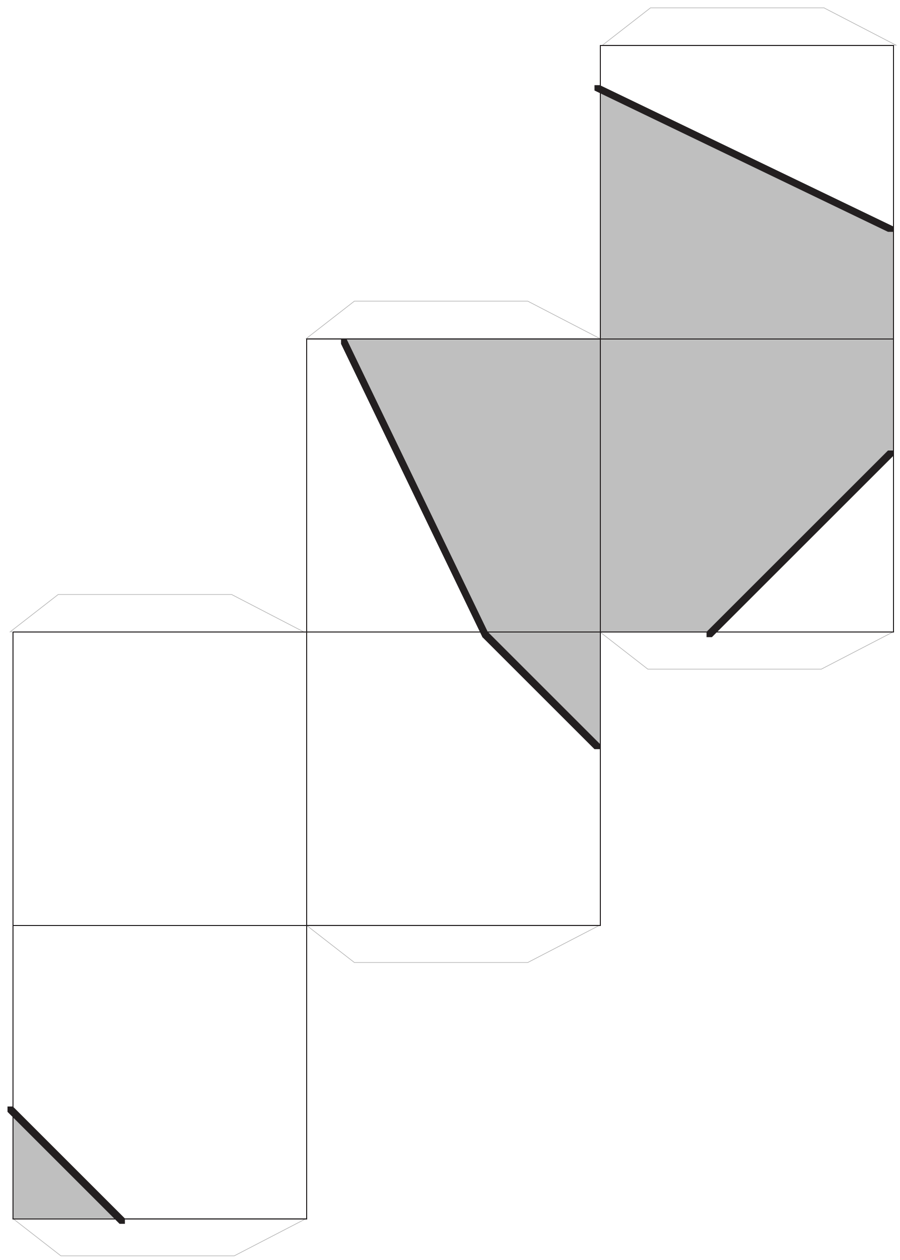
i. A trapezoid: Slice off one edge of the cube with a plane that is not parallel to that edge.

It is impossible to make any polygonal cross section with more than six sides from a cube because that would require the plane to intersect more than six faces, which cannot happen.



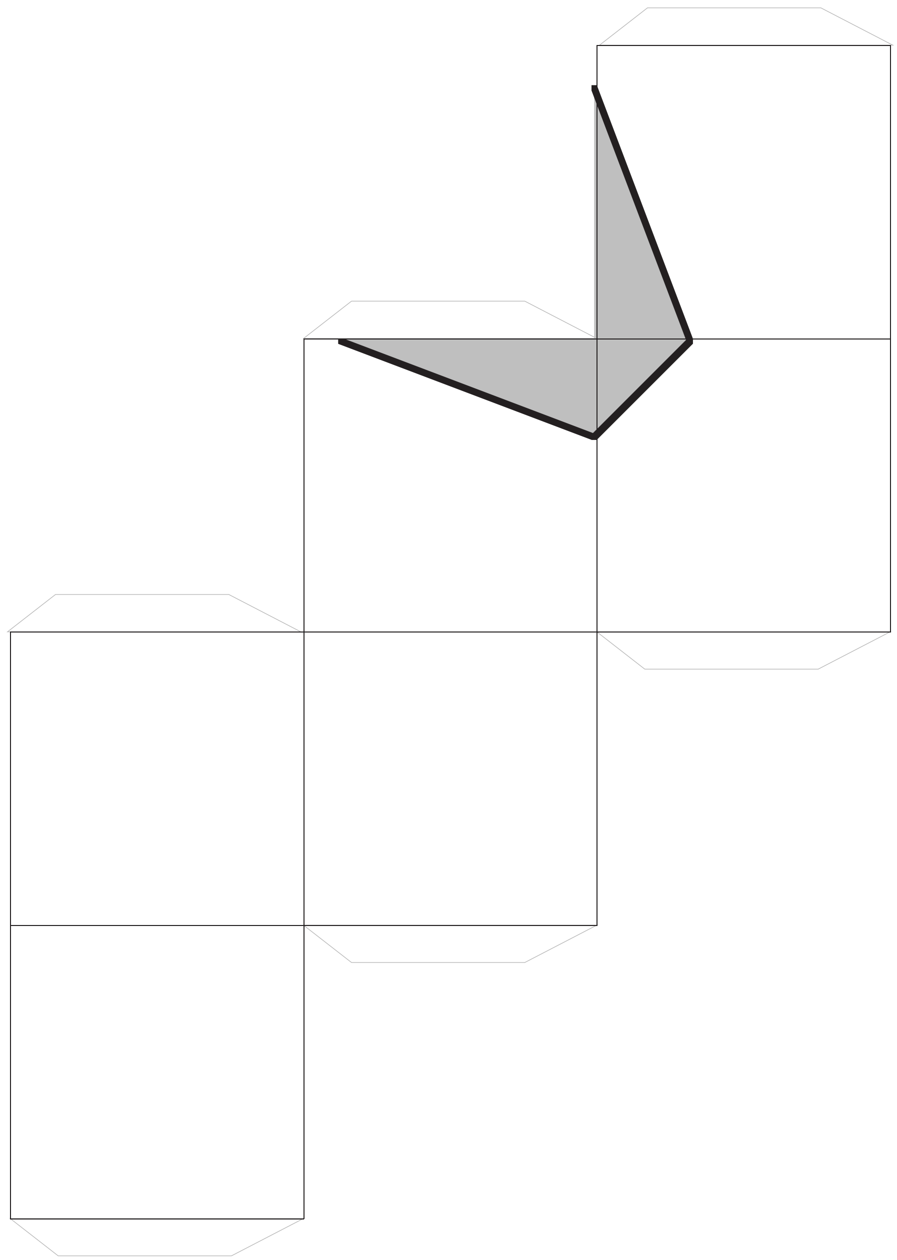
Copy this page. Cut out your copy, fold at all solid lines, and tape tabs inside the shape. Two of these make up a sliced cube, with a hexagonal cross section.

PROBLEM SOLVING IN GEOMETRY *(continued)*



Copy this page. Cut out your copy, fold at the thin lines, and tape tabs inside the shape. A slice placed through the dark lines will separate the cube into two parts, separated by a pentagonal face.

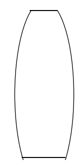
PROBLEM SOLVING IN GEOMETRY *(continued)*



Copy this page. Cut out your copy, fold at the thin lines, and tape tabs inside the shape. A slice placed through the dark lines will separate the cube into two parts, separated by an isosceles triangular face.

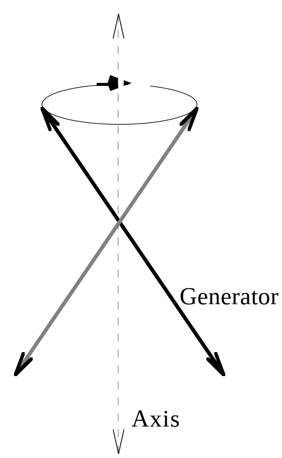
Problem 19 *(Student page 6)* All cross sections of a sphere are circles. The size of the cross section depends on how close the plane cuts to the center of the sphere.

Problem 20 *(Student page 6)* From a cylinder, you may get circles, rectangles, and ellipses, and various "truncated ellipses" like the one below.



Problem 22 *(Student page 7)* There are many possible approaches. One way is to think about the inscribed rectangles of Problem 12 and work from there. In a square, the diagonals must be congruent (diameters of a circle will do) and must be perpendicular bisectors of each other. So connecting the endpoints of perpendicular diameters in a circle will create a square.

Problem 23 *(Student page 7)* Think of the cone as the surface generated by a line (a generator) that rotates about another intersecting line (the axis of the cone).



Vocabulary: ellipse, parabola, hyperbola, cross sections of cone (conic sections)

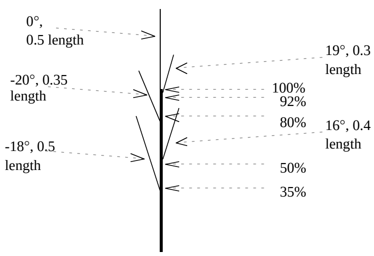
A plane perpendicular to the cone's axis cuts circular cross sections. If you tilt the plane away from the perpendicular, the cross section elongates to an ellipse. As the plane tilts farther, the ellipse lengthens, until the cutting plane is parallel to the

generator. At that point, the cross section opens up and becomes a parabola. If the cutting plane is tilted farther, it cuts both branches of the cone, creating hyperbolic cross sections.

Problem 24 *(Student page 7)* A square pyramid can be cut to make a variety of cross sections, including triangles, quadrilaterals, and even pentagons.

A "Quadrilateral Inequality"!

Problem 25 *(Student page 7)* The sum of any three sides must be greater than the fourth, or the three sides won't fit.



Problem 26 *(Student page 8)* Five limbs grow from the trunk. The lowest limb is 35% of the way up the trunk, half the trunk's length, and growing at an angle of 18° to the left of (or counterclockwise from) the trunk's direction of growth (which is straight up). The next branch grows on the right side of the trunk, halfway up the trunk, angled to the right 16°, and 0.4 the length of the trunk. The diagram shows a plan for the tree visually; words, like the ones above, express the same plan in informal language; a formal (computer) language can also describe the same plan.

Here is how that plan can be written in a computer language. Notice how the indented parts of the computer algorithm encode the same information that the pictorial plan does.

```
to Tree :grwth :sz                 The tree algorithm requires two inputs—the
                                   amount of growth, and the length of the
                                   trunk.
 if :grwth = 0 [stop]
 setroot
  forward :sz * 0.35               The first branch begins 35% up the trunk.
  left 18 tree :grwth – 1 :sz * 0.5 right 18
  forward :sz * 0.15               The next branch begins 50% up the trunk.
  right 16 tree :grwth – 1 :sz * 0.4 left 16
  forward :sz * 0.3
  left 20 tree :grwth – 1 :sz * 0.35 right 20
  forward :sz * 0.12
  right 19 tree :grwth – 1 :sz * 0.3 left 19
  forward :sz * 0.08
  tree :grwth – 1 :sz * 0.5
 return.to.root
end
```

PROBLEM SOLVING IN GEOMETRY (continued)

One can understand how the main body of the program encodes the rule for "tree" growth without understanding these "bookkeeping" procedures.

To run this program in Logo, you need two more procedures to keep track of *where* the drawing is taking place—whether at the trunk, or limb, or branch.

```
to Return.to.root
 pu
 seth first first :root
 setpos last first :root
 pd
 make "root butfirst :root
end

to SetRoot
 if not namep "root [make "root []]
 make "root fput (list heading pos) :root
end
```

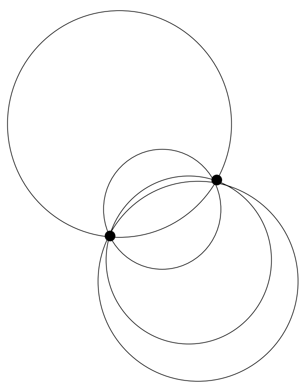
You can get an 80-sized tree by typing `Tree 3 80`.

PICTURING AND DRAWING

Problem 2 *(Student page 11)* Try "holding" two imaginary points in the air as you picture this situation.

a. Only one straight line passes through two points.

b. Do you see only one circle through the two points? Or do you see that a circle through the two points can lie on different planes? Even in a single plane, there are infinitely many, the smallest of them being the one whose diameter is defined by the two points.



Several circles through the same two points

c. There are at least two very reasonable answers to this question. Many different (kinds of) triangles have one side 1″ long, but only *one* (kind of) square can have a 1″ side. From that point of view, there are only two squares, one with the two points defining a side and one with the two points defining a diagonal. But there is also another sensible point of view. Squares can lie in different planes through the two points, so there is an infinite number of them. Both ways of thinking are used in geometry.

d. Exactly three different cubes of different sizes can use the two points as vertices. The points may be adjacent corners of a face of the cube, they may be opposite corners of a face, or they may be at opposite ends of one of the cube's diagonals.

Problem 5 *(Student page 12)*

a. The shadow of a square on a flat surface can be square, nonsquare, or even nonrectangular. It depends on the relationship between the plane of the square and the plane of the surface on which the shadow lies and also on the distance and nature of the light source.

You may find helpful light sources in your school's science lab or a photographic darkroom, or you may use a slide projector or overhead projector.

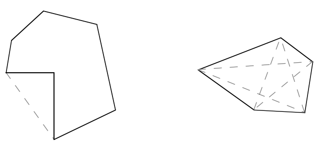
Very different results come from diverging light sources (nearby point sources of light) and "parallel" light rays from strong, direct sunlight (a "point" source of light that is "infinitely" far away). Without special focusing, any (point) source of light will cast a larger shadow as the object moves closer to it. Because the sun is so far away, the effect of such changes of distance are not noticeable. As a result, in sunlight, the opposite sides of the shadows of squares are equal even if the square is tilted in a way that brings some parts of it closer to the sun than others. Thus, the shadows of squares (in sunlight) must be parallelograms. Shadows generated by nearby sources of light can be more general quadrilaterals.

b. Which of the given shapes can cast a square shadow? Some are relatively easy to figure out, and others are extremely difficult for virtually everybody.

Shapes 3 and 5 cannot cast square shadows.

"Planar" means "in a plane," a two-dimensional figure.

In a *concave* polygon, at least one diagonal falls outside the polygon. In a *convex* polygon, all diagonals fall inside the polygon.



concave *convex*

Shapes 1, 7, and 8 can cast a square shadow. What about 6 and 2?

When making cross sections, you can get triangles and pentagons by cutting through the square base of the pyramid. This doesn't extend to shadows; that is, the triangle and the pentagon cannot cast square shadows. Why?

Vocabulary: equilateral, isosceles, scalene

Here is one way to think about it: If a two-dimensional shape casts a shadow, the shadow must have the same number of corners as the shape. Why is the number of corners invariant? Experiment with two straws taped together at an angle. The only way to get the angle (and thus the vertex) not to appear in the shadow is to have the shadow appear straight. To do that, the plane in which the angle lies must pass through the light source. A planar shape would cast no shadow (the "shadow of a line segment") if lit this way.

Experimentation will also show that concave planar shapes (like shape 4) cannot cast convex shadows, so shape 4 cannot cast a square shadow. In the language of invariants, concavity and convexity are "preserved" in shadows of planar shapes.

Shape 1 obviously *can* cast a square shadow, and a little experimentation shows how shape 8 can, under the right conditions. Some people "see" that shape 7 can, and most people remain uncertain about shapes 6 and 2. In all three of those cases, it is extremely difficult to "explain your answers."

Another way to think about this problem is to relate it to Problem 24 on page 21. Imagine a square (the "shadow") flat on the floor and a point (the "source of light") anywhere above the floor (though not necessarily directly above the "shadow"). Connect each vertex of the square to the light with a string. The shape that you've outlined is a kind of square pyramid. Any plane that passes through the four strings defines a cross-sectional shape. *That* shape in *that* position will cast *that* square shadow if lit by *that* light.

Problem 6 *(Student page 13)* With any light sources, an equilateral triangle may cast equilateral, isosceles and scalene shadows. In the sun, a circle may cast circular or elliptical shadows. With nearby light sources, circles can cast egg-shaped (oval) shadows, too.

Problem 7 *(Student page 13)* Each of the following letters, if it is drawn simply enough, can be cut into matching parts: **A, B, C, c, D, E, H, I, i, K, L, l, M, N, O, o, S, s, T, U, V, v, W, w, X, x, Y, Z, z.**

Problem 8 *(Student page 13)* Here are some possible observations: "The mouth is one third the width of the head, while the eyes are one fifth and the nose is one ninth." "The eyebrows are placed roughly halfway from top to bottom." "The height of the face is roughly three fifths the height of the head." "The whole head, by measurement, is about 25% taller than it is wide."

Problem 9 *(Student page 14)* It is usually easiest to start from general shapes and work to details. Start with the overall shape of the head. Is it more square or more

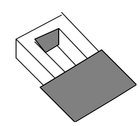
oblong? If oblong, roughly what are its proportions? From that rectangular frame, round it out. Is the bottom the same width as the top, or is the head more like an upside-down eggplant?

There is no special rule about the grid lines. After you have overall measurements—like the width and height of the head and, say, the height of the face and the placement of one significant facial feature—just pick some grid lines that help you "see" the other relationships. They might be regularly spaced, like the ones shown above Problem 8, or might pass through particular features (like the eyes) to help you determine the size or placement of the mouth or ears. Dividing the head into thirds and fifths helped *this* artist with *this* face, but other schemes might be more helpful in your drawings.

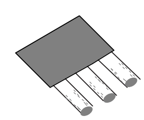
After you've finished the next two investigations, try again to draw a face or something else that requires "seeing" in this new geometrical way.

Problem 11 *(Student page 15)* There are many ways to describe what's strange about the picture. Here is one.

One end of the picture seems to show a rectangular arch, a structure with two rectangular legs extending from a rectangular top.



The other end of the picture seems to show that the thing has three legs, not two, and that the legs are rounded rather than rectangular.



Either end makes perfect visual sense. Together, they do not.

Problem 13 *(Student page 17)* Parallel faces would normally be shaded the same way because they receive sunlight at the same angle. They get different shading only when other objects change the light that they receive (by casting shadows on them, for one example), or when they are different colors or textures to begin with.

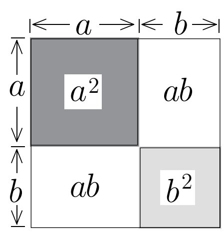
PICTURING AND DRAWING *(continued)*

Problem 14 *(Student page 18)* Shading helps the illusion by helping the eye to believe it is following a single smooth flat surface.

Problem 15 *(Student page 18)* As in Problem 11, there are many ways you might describe what's "wrong" with the picture. In the first of the three drawings shown in this problem, the white side seems to be in front, while in the second it becomes a side face, and in the third it becomes an inner face.

Problem 17 *(Student page 19)* These pictures tend to "flip," with the lighter color planes appearing sometimes as top faces and sometimes as bottom faces, so that each drawing suggests two different pictures. If you have trouble *seeing* the pictures these two ways, try viewing them upside down.

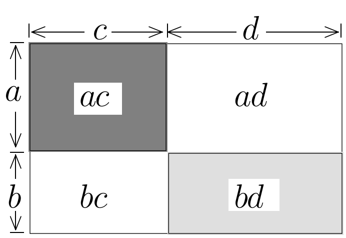
Problem 18 *(Student page 19)* How does the picture illustrate the algebraic identity? A good answer will contain explanations of why the area of the overall square is $(a+b)^2$, how a^2, b^2, ab, and $2ab$ are related to the figure, and how adding a^2, b^2, and $2ab$ relates to the picture. Here is one possibility:



$(\mathbf{a}+\mathbf{b})^2 =$
$\mathbf{a}^2 + 2\mathbf{ab} + \mathbf{b}^2$

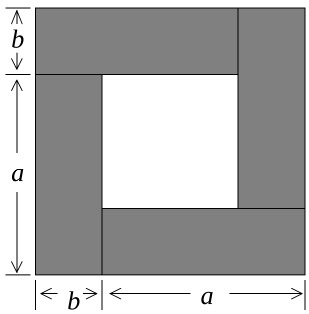
> The large square measures $(a+b)$ on a side, so its area is $(a+b)^2$. The large square is divided into four regions: two squares and two other rectangles. One square has sides of length a, so its area is a^2. The other square has sides of length b, so its area is b^2. The rectangles have the same dimensions: one pair of sides length a and the other pair length b, so each area is ab. If we add the areas of the four regions, we get the area of the large square: $(a+b)^2 = a^2 + 2ab + b^2$.

The same *kinds* of information should be included when explaining how the rectangle on the right illustrates the algebraic identity $(a+b)(c+d) = ac + ad + bc + bd$. Here's one explanation:



$(\mathbf{a}+\mathbf{b})(\mathbf{c}+\mathbf{d}) =$
$\mathbf{ac} + \mathbf{ad} + \mathbf{bc} + \mathbf{bd}$

> The large rectangle measures $(a+b)$ on one side and $(c+d)$ on the other, so its area is $(a+b)(c+d)$. The large rectangle is divided into four smaller rectangles. One rectangle has sides of length a and c, so its area is ac. Another rectangle has sides of length b and c, so its area is bc. A third rectangle has sides of length a and d, so its area is ad. The last rectangle has sides of length b and d, so its area is bd. If we add the areas of the four regions, we get the area of the large rectangle: $(a+b)(c+d) = ac + ad + bc + bd$.



$(a+b)^2 - 4ab = (a-b)^2$

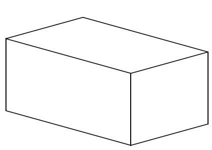
Problem 19 *(Student page 20)* A good explanation should include these four parts:

- The sides of the large square are $(a+b)$, so its area is $(a+b)^2$.
- The four gray rectangles in the figure are rectangles with sides of length a and b, so each has area ab.
- The unshaded central region is a square with sides of length $(a-b)$, so its area is $(a-b)^2$.
- Another way to get the area of the central region is to take the area of the large square and subtract the areas of the four gray rectangles. This must be the same as the first value for the area (computed by using the lengths of the sides), so $(a+b)^2 - 4ab = (a-b)^2$.

Problem 20 *(Student page 20)* The teacher drew a graph, with a horizontal *time* axis and a vertical "*terror*" axis. The graph shows the teacher's terror increasing fairly rapidly, plummeting suddenly, and then gradually tailing off, presumably as the teacher relaxes after the test.

There is no real way to tell where, on the graph, the exam was taken. Perhaps it took place just as the terror is nearing its peak, and all the buildup before is anticipatory fear. Perhaps the exam started where the graph starts and ended at the peak of terror, with the teacher's discomfort growing throughout the exam. It is equally possible that the anxiety all built up while the teacher was studying for the exam, but dropped the moment the teacher saw the exam and realized that it would be easy for her.

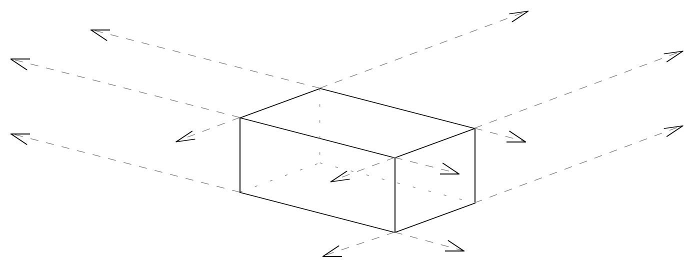
Problems 22–23 *(Student page 20)* Here are some of the specialized terms used in the last few pages of the Student Module: base of a prism, binomial, corresponding corners, cube, equilateral, Escher, fractal geometry, hexagon, nonrectangular, non-square, optimizing, parallel faces, parallel lines, parallelogram, pentagonal, polygon, prism, pyramid, quadrilateral, segments, vertical, and wire frame drawing.

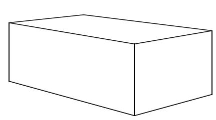


"Parallel perspective"

Problem 24 *(Student page 21)* In mathematics textbooks (elementary through high school) prisms are often drawn according to the method (called "orthogonal projection" or "parallel perspective") described in Steps 1 through 6 on pages 16–17 in the Student Module. Just as the sun's great distance preserves parallels in

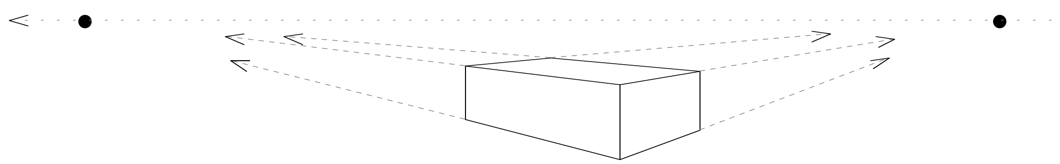
shadows, "parallel perspective" preserves parallels. It shows how an object would look if photographed from very far away and then enlarged enough to be seen.

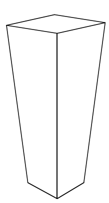




"True perspective"

Pictures of prism-like objects in magazines or newspapers are typically *not* drawn the same way, but are rendered in "true perspective," the view one gets at one's usual, relatively-close vantage point on objects. In such views, verticals in the real-life scene remain vertical in the picture, but other lines that are parallel to each other in the 3D scene converge in the picture.





"Three-point perspective"

Occasionally, for special effects, magazines will show how things look when a camera lens takes in a wider (or taller) view than the eye can normally see. In some of these views, straight lines will seem to curve, as they do in the images seen on the curved passenger-side outside mirrors on buses and some cars.

Problem 25 *(Student page 21)* This problem reviews the Triangle Inequality.

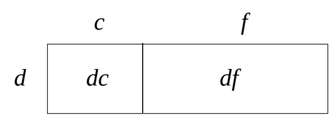
a. 2″, 7″, and 5″? No; 2 + 5 is not greater than 7.

b. 3″, 7″, and 5″? Yes

c. 2″, 7″, and 3″? No; 2 + 3 is less than 7.

d. 4″, 4″, and 4″? Yes; equilateral!

Problem 26 *(Student page 21)*



$$d(c+f) = dc + df$$

Problems 28–29 *(Student page 21)* One way to look at it is that the figure is impossible as it appears, and so no “planes” are shown! Another way is to respond to the *intent* of the artist, and claim that three “planes” are pictured in the sketch (white, grey, and dotted), while another three “planes” are hidden. Similar discussions can come up with the other objects listed.

Problem 29 *(Student page 21)* The *base* is called “base” because it is the basis for the 3D shape.

This approach gets an answer, and perhaps even a general conjecture, but may not give enough insight into *why* that answer works.

Problem 30 *(Student page 22)* There are several good strategies. One kind of experiment starts by counting the regions formed when only one line cuts the plane. Then, by carefully adding more lines one at a time, a pattern in the number of regions begins to emerge. Experimentally, one can find a maximum of 16 regions for five lines.

One student described this as “the more different the lines are, the more regions.”

Another approach is to see how changes in the arrangement of a given number of lines can increase or decrease the number of regions. Parallel lines minimize the number of regions. Getting the maximum, therefore, involves avoiding parallel lines. Similarly, when three lines cross at the same point, one fewer region is produced than when all lines cross in pairs. This leads to a conjecture that when all lines cross each other (and no three lines cross at the same point), the plane is divided into the maximum number of regions. More ideas can come from working Problem 32.

Problem 31 *(Student page 22)* This is an extremely challenging problem, worth working on for a while, putting aside, and coming back to periodically throughout the year. Problem 32, which is in many ways trivial, offers a way of thinking about these questions. Incidentally, five planes will divide space into a maximum of 26 regions; the maximum for six planes is 42 regions. But the *real* task is to understand why!

Problem 32 *(Student page 22)* Just as “the more different the lines are the more regions,” the more different the points (in this problem) are, the more regions there

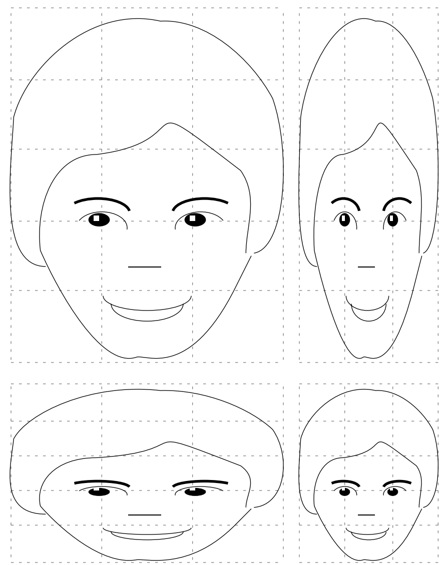
will be. But there is only one way for points to be different, and that is in location! Points that coincide divide the line into fewer regions than distinct points. We see that n distinct points will divide a line into $n + 1$ regions.

The table below summarizes the maximum number of regions that can be created in the ways described in Problems 30 through 32.

Number of objects dividing an object of one higher dimension	0	1	2	3	4	5	6	…
Number of regions created by points on a line	1	2	3	4	5	6	7	…
Number of regions created by lines on a plane	1	2	4	7	11	16	22	…
Number of regions created by planes in space	1	2	4	8	15	26	42	…
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Problem 33 *(Student page 23)* Once you make a drawing on a grid, you can rescale or distort the drawing by using an altered grid. For a good enlargment or reduction, all of the squares in the grid should remain squares. If the grid is changed

by different amounts in each direction, the picture will not be a “good” enlargment or reduction.



Drawing and Describing Shapes

Problem 1 *(Student page 24)* The shadow of a soup can may be a circle or ellipse, a rectangle, or a salami shape, depending on how the can is held and how it is lit. A cube can cast a square shadow and also a variety of rectangular, pentagonal, and hexagonal shadows.

Problem 2 *(Student pages 24–25)* A solid that casts circular or rectangular shadows, depending how it is lit, could be a cylinder.

Problem 3 *(Student page 25)* A solid that casts circular or triangular shadows, depending how it is lit, could be a cone.

Problem 4 *(Student page 25)* The tip of a screwdriver (regular, not a Phillips head) can cast shadows that are triangular, circular, or rectangular (or other shapes), depending on how it is lit. For very difficult problems like this, it helps to think of one direction at a time or to think of how you can combine the objects from Problems 2 and 3 into a single object.

Problem 5 *(Student page 26)* The expression "turn right" is not really specific enough, but if one assumes that "turn right" (where not otherwise specified) means "turn 90° right," then following the directions makes a rectangle and returns one to one's original position, facing north.

Problem 6 *(Student page 26)* In Problem 4, the focus was on features. Names (square, circle, triangle) were used to specify the features, but the *features* were used to specify the 3D object. In Problem 5, the focus was on a recipe. The resulting shape was *described* by name (rectangle) but was *specified* by recipe.

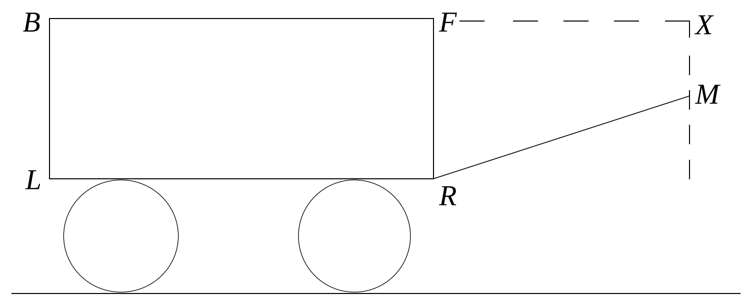
Solutions

Investigation **1.4**

Student Pages 27–28

DRAWING FROM A RECIPE

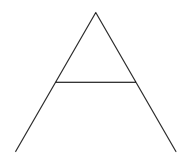
Problem 1 *(Student page 27)* Steps a–h give instructions for drawing a wagon.



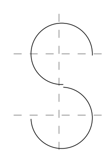
Vocabulary: median, concurrent, centroid

Problem 2 *(Student page 27)* The segment connecting a vertex with the opposite midpoint is called a *median*. The three medians of any triangle are *concurrent* (meet at a single point). The point at which they meet is called the *centroid*.

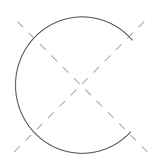
Problem 3 *(Student page 28)* These instructions form a capital letter A.



Problem 4 *(Student page 28)* These instructions form a letter S.



Problem 5 *(Student page 28)* These instructions form a letter C.



Drawing from a Recipe (continued)

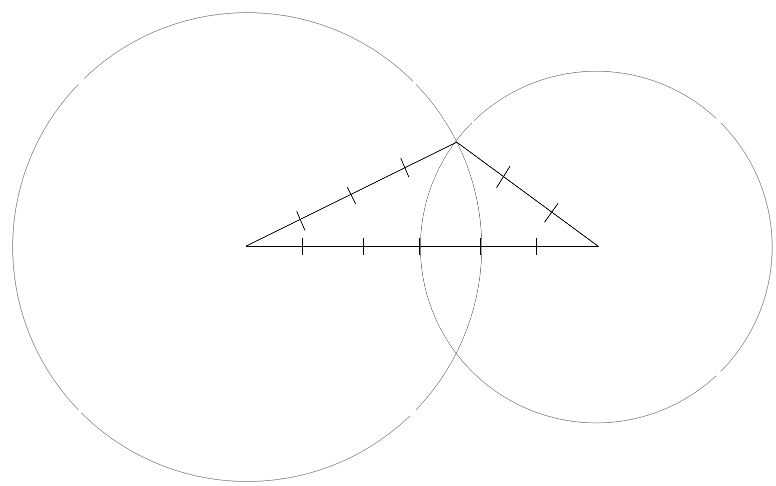
Problem 7 *(Student page 28)* It is relatively easy to describe a picture well enough to pick it from among others; it is quite a bit harder to describe how to draw it. Compare the two descriptions below.

- It looks a bit like the head of a strange insect: two enormous eyes, a kind of beak below, and three hairs or feelers coming out of the top.
- Draw a square with a horizontal base. Draw a circle the same height as the square, and centered on the right side of the square so that the square's side is the circle's diameter. Draw an identical circle the same way, centered on the left side of the square. Mark two points on the bottom of the square, each about a quarter of the way in from the sides. Using the segment between those two points as one side, draw an equilateral triangle outside the square (in other words, pointing down from the square). From the midpoint of the top of the square, very lightly draw a perpendicular segment extending upward, about half the length of the top of the square. (You will erase this segment later.) Now, using three other segments of roughly the same length drawn from the same midpoint of the top of the square, subdivide the right angle on the right-hand side into four roughly equal-length parts. Finally, erase the vertical segment from the top.

There are often many choices for describing a picture. Sometimes it is especially convenient to describe parts of the picture with the aid of "construction lines" even though these extra lines must later be erased.

CONSTRUCTING FROM FEATURES: PROBLEM SOLVING

Problem 1 *(Student page 29)* The picture below, showing a small version of this triangle with two circles as "construction lines," suggests one way to look at the problem, but being allowed only the use of one ruler, one piece of paper, and one pencil makes this problem challenging.



Here is one method to approximate the "compass technique" shown above: Using your ruler, draw a six-inch segment on your paper. Use the ruler to mark off several places that are three inches away from one endpoint of the segment. You will have several dots along the circle with three-inch radius centered at that endpoint. Now use the ruler to find a place that is four inches from the other endpoint and that seems to cross your path of dots. Check that the distance from this point to the first endpoint is three inches. If so, it is your third vertex. If it is off a little, you can refine it by marking more points that are three inches from your first endpoint and trying the process again.

Problem 2 *(Student page 30)* This construction in 2c is impossible: You cannot make a triangle unless the sum of *any* two sides is greater than the remaining side. (the converse of the Triangle Inequality again!) In this case, $3 + 4 < 8$. All the other constructions are possible.

Problem 3 *(Student page 30)* When three sides are known, a triangle is specified completely. If all work were accurate, everyone's corresponding triangles would be the same; their angles would therefore be the same. In practice, the triangles will not be identical. Ones that are very far off may have errors, but slight variation is to be expected. How alike do results need to be before we suspect that something special is going on?

Problem 4 *(Student page 30)*

THEOREM ***Angle Sum in Triangles in a Plane*** **(proved later)**

The sum of the measures of the angles in a triangle in the plane is an invariant: for any triangle, the angle sum is 180°.

Because of inaccuracies in measurement, many triangles you draw may have angle sums that are close to, but not exactly, 180°.

Problem 5 *(Student page 31)* This problem is (and probably feels!) quite different from the previous one. The triangles seem to squirm about more and are harder to pin down.

The triangles in 5b and 5d cannot be drawn on a plane: their angle sum is too large. All the others can be drawn in a plane.

Problem 6 *(Student page 31)* Size varies, but proportion is invariant. The triangles that students create for 5a, for example, will *not* all be identical, but the differences should only be a matter of scale: the angles should match, and the measured ratios should be very close. These ratios would be absolutely identical if measurement and construction could be perfectly precise, but such precision is possible only with the mathematical tools of one's mind, and not with the tools in one's hands.

Problem 7 *(Student page 31)*

THEOREM ***SSS*** **(not proved here)**

Three sides uniquely determine a triangle.

Three angles do not uniquely determine a triangle but do determine its proportions, which is equivalent to saying that they determine its "shape."

Problem 8 *(Student page 31)* Here is one way to create a half-sized triangle without measuring: Fold two sides of the original triangle to find their midpoints, and then draw the "midline" (also called a "midsegment") connecting these midpoints. Each of the three sides of the resulting small triangle is half the length of a side of the original triangle. (The *area* of the small triangle, however, is one fourth that of the original, which can be seen by laying copies of the smaller triangle on the larger.)

Problem 9 *(Student page 32)* The first side is drawn at the specified length. Then a ruler is used to represent each of the other lengths, and the two rulers are swung

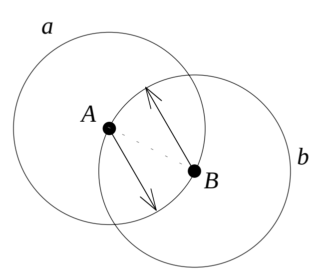
towards each other until they meet. Their ends describe an arc, performing the function of a compass.

This method is precisely the one that is used with compass, straightedge, and three specified lengths.

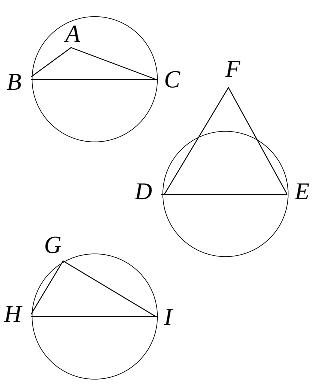
This is essentially the same as the solution to Problem 1 in this investigation because this is essentially the same problem.

Problem 10 *(Student page 32)* To construct a triangle with one ruler and no compass, begin by drawing the first side at the first specified length. Then, use the ruler to represent a second length. Place the ruler at one endpoint of the first side, swing it through an arc, and trace its path with a pencil. (This operation is quite clumsy, which is why compasses are better!) Then move the ruler to the other endpoint of the first side and place it to meet the path of the arc.

Problem 11 *(Student page 33)* Two circles that pass through each other's centers must have the same radius. (If the two centers are A and B, then each circle must have a radius equal in length to AB.) Set a compass at any radius and draw a circle. Then place the point of the compass anywhere on your circle and, without changing the radius of the compass, draw a second circle.



Problem 12 *(Student page 33)*



a. Vertex A is inside the circle.

b. Vertex F is outside the circle.

c. Vertex G is on the circle.

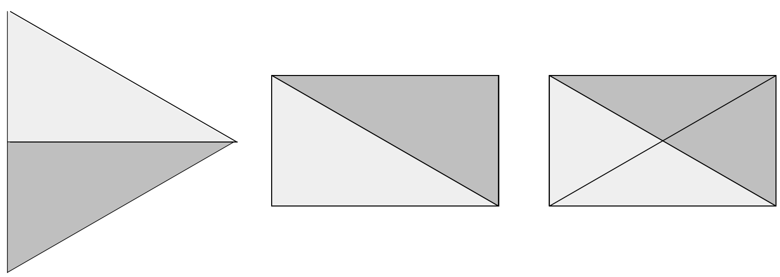
For Discussion *(Student page 33)* The result with the right triangle recalls the angle invariant in Problem 11 in Investigation 1.1, "inscribing an angle in a semicircle." The new experimental data (Problem 12) still does not prove the conjecture, but it does lend more support. This problem shows not merely another case of a 90° angle falling on the circle, but also an angle greater than 90° falling inside, and an angle less than 90° falling outside the circle. This is *strong* support but should not be confused with proof.

Problem 13 *(Student page 33)*

a. It *is* possible to construct a rectangle whose diagonal is twice the length of one of the sides. Here is one way of reasoning: Draw a rectangle whose diagonals are shorter than twice the shortest side; then draw one whose diagonals are longer than twice the shortest side. Now imagine completing this spectrum with a *continuum* of rectangles whose diagonals progress from shorter to longer. Somewhere in that continuum must be a rectangle whose diagonals have the required length.

Knowing that such rectangles exist is not the same as knowing how to construct one. *That* is harder, and may require facts you can't yet assume.

Here's one construction (without an explanation of why it works): Cut an equilateral triangle in half from a vertex to the opposite side. Reassemble the pieces into a rectangle. The diagonal of the rectangle is the length of the full side of the equilateral triangle, and the short end of the rectangle is half that length.



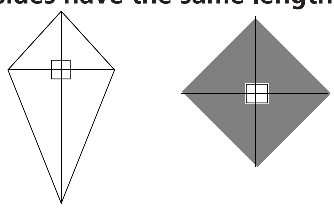
b. A rectangle in which a diagonal is the same length as one of the sides is impossible. The diagonal divides a rectangle into two right triangles, in which the diagonal serves as the hypotenuse.

THEOREM *Hypotenuse Longest*

The hypotenuse of a right triangle is longer than either of the legs of the triangle.

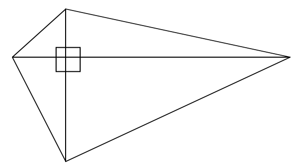
Why? Because the shortest path from a point to a line runs perpendicular to that line. So the shortest path from an endpoint of the hypotenuse to the side opposite that point runs along a leg of the triangle, not along the hypotenuse.

Kites **have two pairs of congruent, adjacent sides.** ***Rhombi*** **(or** ***rhombuses*****) are really special kites: all four sides have the same length.**



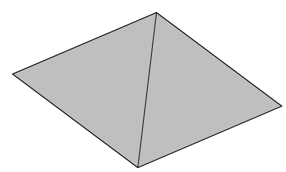
kite *rhombus*

c. To construct a quadrilateral whose diagonals are perpendicular to each other, draw the two diagonals first and then connect their endpoints. This kind of quadrilateral includes, as special cases, kites and rhombi (including squares), but also includes many other shapes that have no special name. Here is an example:



d. A rectangle whose diagonals are perpendicular to each other is a square.

Problem 14 *(Student page 33)* There is only one quadrilateral with at least one 60° angle and sides that are all the same length: a rhombus consisting of two equilateral triangles joined at a side.



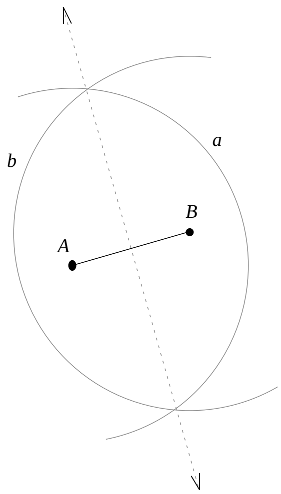
Problem 15 *(Student page 34)* The simplest approach is to use symmetry by folding.

Fold the segment so that its endpoints lie on top of each other. This matches its two halves exactly; the point that separates these two parts is the midpoint. In fact, all points on the fold line are equidistant from the two endpoints of the segment. This is easiest to see while the paper is folded. Any point on the fold is the same distance from each of the two original endpoints, because they now lie at the same place! The fold line is also perpendicular to the segment (and therefore is the perpendicular bisector), which can be shown by matching angles around it and showing that they are equal in measure and that adjacent ones sum to 180°.

THEOREM ***Perpendicular Bisector***

All points on the perpendicular bisector of a segment are equidistant from the two endpoints of the segment.

CONSTRUCTING FROM FEATURES: PROBLEM SOLVING (continued)

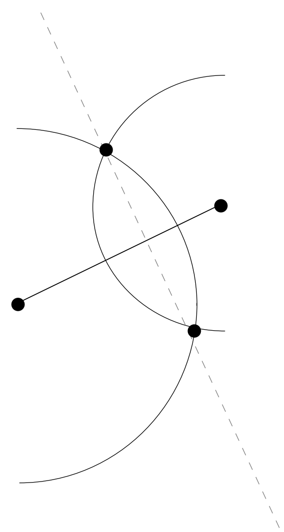


Alternatively, the midpoint can be constructed with compass and straightedge. Swing two equal-radius arcs (with radius R greater than half the original segment), one centered at each endpoint of the segment. The two points of intersection of the arcs are each R units from both endpoints of the segment. Perhaps the easiest way to show that all the points along the line connecting these two intersection points are *also* equidistant from both endpoints of the segment is to fold along that line (equivalent to the paper folding solution). When the two endpoints of the segment are superimposed in this way, one may travel the same distance from any point along the fold to each endpoint of the segment.

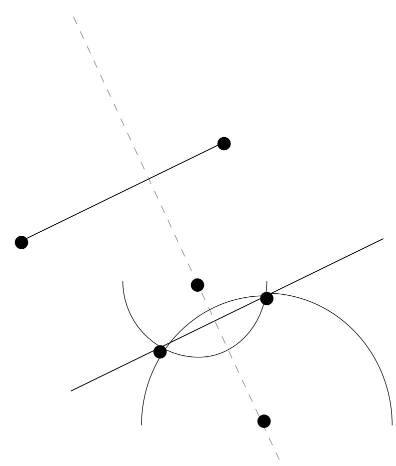
Problem 16 *(Student page 34)*

a. For a perpendicular line, fold the paper as in Problem 15, though the endpoints of the segment do not necessarily have to meet. To construct an arbitrary perpendicular with compass and straightedge, swing two arcs, as in the construction of the perpendicular bisector, but without assuring that the two arcs are equal.

b. For constructing a parallel line, construct a perpendicular to the perpendicular.

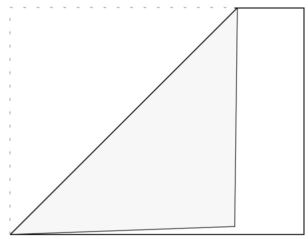


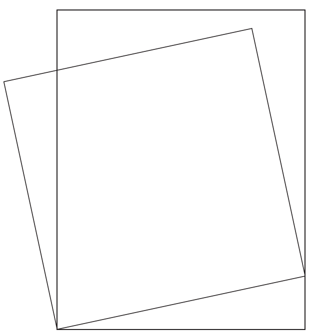
Perpendicular line



Parallel line

Problem 17 *(Student page 34)* To create a square from a rectangular sheet of paper, fold the shorter edge of the paper along the longer one. Cut off the excess paper.





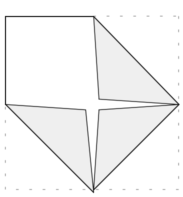
Why is this the largest square possible? Think about the sidelengths. To get a length greater than $8\frac{1}{2}$ inches, we would have to rotate the square so that its sides don't coincide exactly with the paper's sides, but then (because we still need 90° angles) the square is no longer contained in the paper.

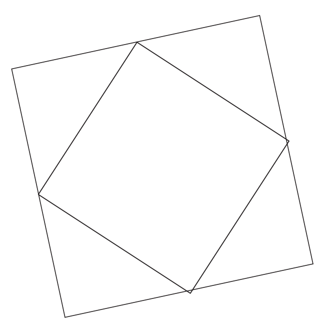
Problem 18 *(Student page 34)*

a. To construct a square with exactly one-fourth the area of your original square, fold the original in half, and then fold in half again in the other direction.

b. To construct a square with exactly half the area of your original square, fold all the corners to meet precisely at the center of the original and crease well. Because the four folded-over corners exactly cover what is left, we see that they and what's covered have the same area—half the area of the original figure. Now unfold. The creased lines form the required square.

Another way to say this is that we connect the midpoints of the sides of the square in order, but that doesn't explain why it works.





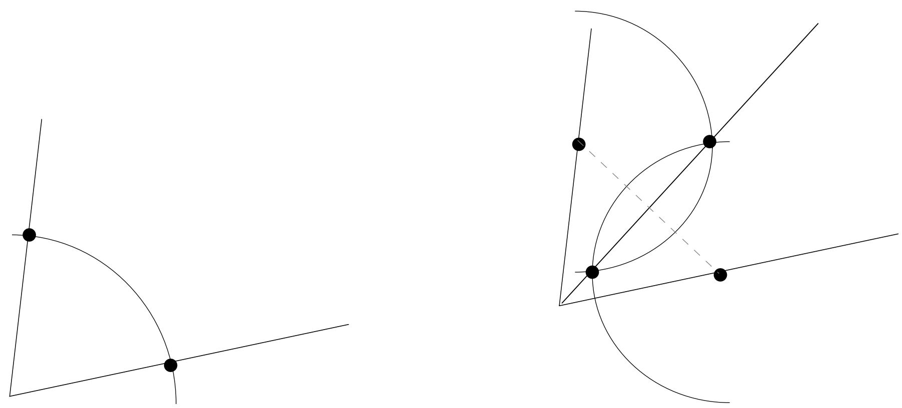
Problem 19 *(Student page 34)* To construct two rectangles with the same area, start with any rectangle and fold it in half in either direction, parallel to one pair of sides.

Problem 20 *(Student page 34)* To construct an angle bisector by folding, fold so that the two angle sides coincide. Analyzing this proves the following theorem:

THEOREM ***Angle Bisector***

All points on the bisector of an angle are equidistant from the two sides of the angle.

To solve this problem with straightedge and compass, center the compass at the vertex of the angle and swing an arc that intersects both rays that form the angle. The perpendicular bisector of the segment that connects those two new intersection points is also the bisector of the angle.

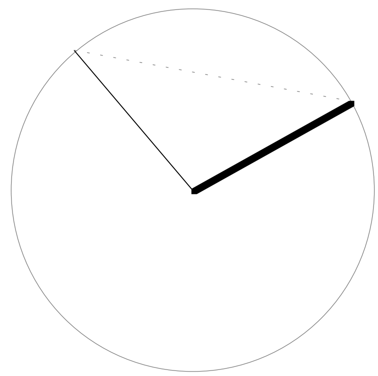


Problem 21 *(Student page 34)*

a. You want to construct an isosceles triangle with a given segment as one of the two *congruent* sides.

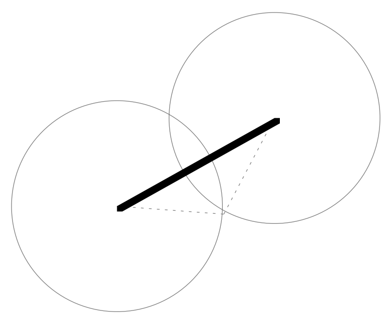
A paper-folding solution: Make fold through one endpoint of the segment, making sure that the folded piece covers the second endpoint, and crease well. At the other endpoint of the segment, poke a pencil point or pin through both layers of paper. Unfold the paper and connect the hole that is not on the segment to both ends of the segment. As long as the original crease was neither perpendicular to the segment nor lying along the segment, this will produce the required triangle.

A compass solution: Construct a circle with radius equal to the given length. Any two radii of this circle which do not form a diameter are two equal-length sides of an isosceles triangle.



b. You want to construct an isosceles triangle whose base is a given segment.

A compass solution: Using one end of the given segment, construct any circle with a radius larger than half that segment. Using the other end of the given segment, construct a second circle with the same radius. An intersection of the two circles is a point that is the same distance from each center.



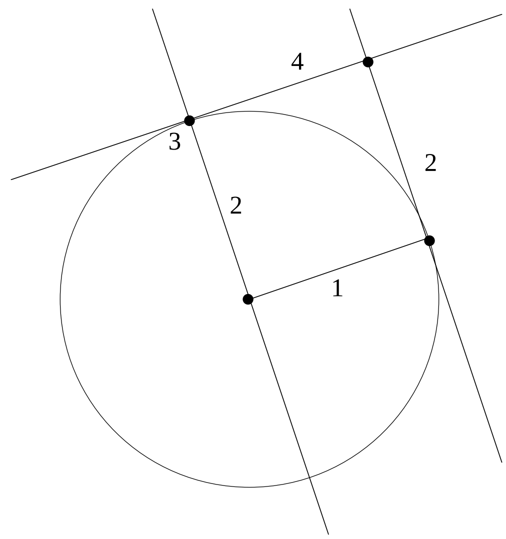
A paper-folding solution: Fold the segment in half. Any point along the crease (the perpendicular bisector) can be used as the vertex of the required triangle.

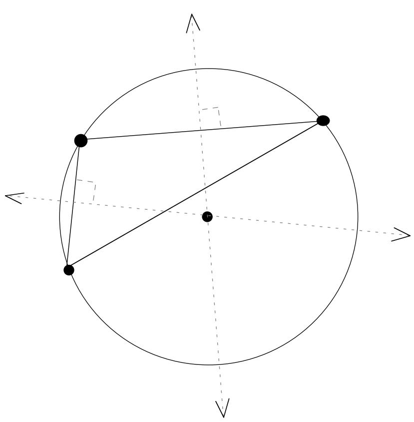
c. You want to construct an equilateral triangle based on your segment. You can use the compass method given above, but make the radius of each circle *equal* to the length of the given segment.

d. You want to construct a square based on your segment. The numbered steps below correspond to the numbers on the construction shown below.

1. Start with your given segment.
2. Construct perpendiculars to the segment through each of the two endpoints.
3. Along one of those perpendiculars, "copy" the length of your original segment. You can do this with a compass or by paperfolding.
4. From the endpoint you just found, construct a perpendicular to that side and finish the square.

To copy the length by paper folding, bisect the 90° angle so that the original segment is lying on top of the perpendicular line. Poke a hole through the other endpoint (not at the folded vertex) that goes through both layers of paper. This hole is the third vertex of the square.





Constructing a circumcircle

Problem 22 *(Student page 35)* For the three points to be on a circle, they must all be the same distance from the center of the circle. All points along a perpendicular bisector are equidistant from the endpoints of the bisected segment. (See Problem 15.) So, for *any* triangle (not just an equilateral one), construct the perpendicular bisectors of two sides, and their point of intersection will have to be equidistant from all three vertices. Center a circle at that point. If it is made to pass through any vertex, it will pass through all three.

THEOREM ***Three Point Circle***

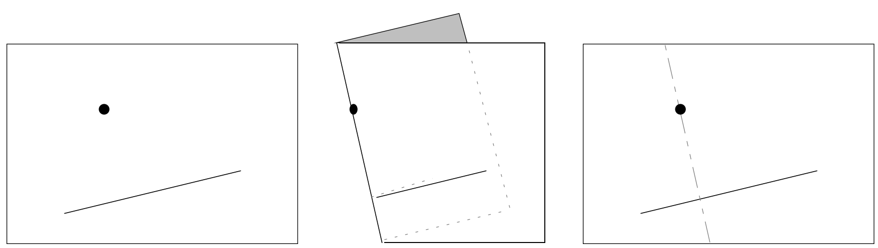
Given any three noncollinear points, it is possible to construct a circle through them.

Problem 23 *(Student page 35)* Here are two of the many possible correct illustrations.

Medians *Midlines*

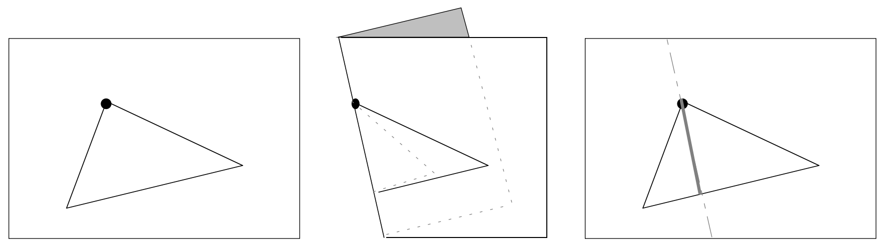
Problem 24 *(Student page 35)* Paper-folding solutions or compass and straightedge solutions can be used. Midpoints (the key to medians and midlines) and angle bisectors have been constructed in earlier problems.

To construct altitudes, one must be able to construct a perpendicular to a given line through a given point. The figure below illustrates a paper-folding method: Fold the line onto itself in a way that allows the fold to pass through the point.

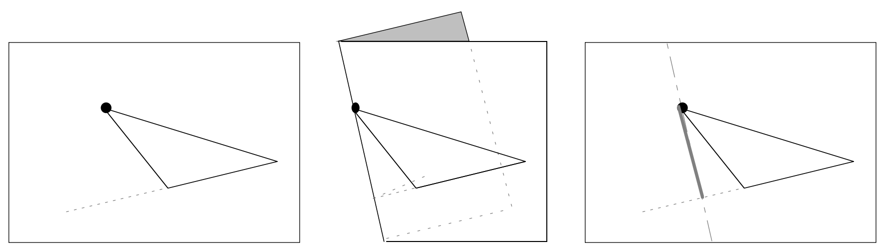


In the case of an altitude of a triangle, the given line contains one side of the triangle (which is not a line, of course, but only a segment, and is called the base) and the

given point is the opposite vertex. The altitude is a segment along the constructed perpendicular and extends from the vertex to the line containing the opposite side.



When the chosen base of the triangle is adjacent to an obtuse angle, the altitude to that base will lie outside the triangle, but the construction methods are essentially the same; you must simply extend the base of the triangle by drawing or folding.



Problem 25 *(Student page 35)* In accurate constructions, the three medians of any triangle will intersect at a single point (called the *centroid*). Likewise, the angle bisectors intersect at a single point (called the *incenter*), and the altitudes (or the lines that contain them) intersect at a single point (the *orthocenter*). The midline construction divides the original triangle into four triangles that are all the same shape and size.

Problem 26 *(Student page 35)* The diagonals of a square have the same length; each of them divides the square in half; they intersect at right angles; together they cut the square into quarters.

Problem 27 *(Student page 36)* Lightly sketching in the medians greatly reduces the amount of work needed to make this sketch, especially if the original triangle is not equilateral.

Problems 28–29 *(Student page 36)* In this investigation, we have used many terms—general vocabulary, as well as the technical vocabulary of mathematics—

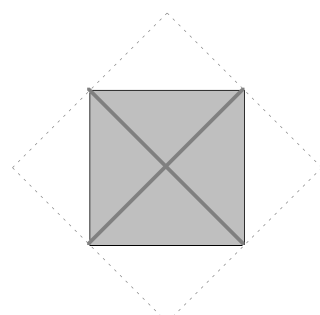
that are new or have specialized meanings. These include acute triangle, altitude (of a triangle), angle bisector, angle sum, arc, bisector, compass, construction lines, construction vs. drawing, diameter, equilateral triangle, good enlargement/reduction, horizontal, infinite, isosceles triangle, median, midline, midpoint, obtuse triangle, opposite vertex, perpendicular, perpendicular bisector, quadrant, radius, ratio, ray, regions, right triangle, ruler, straightedge, symmetry, tangent, tentative conclusion, and unambiguous directions.

Problem 31 *(Student page 36)*

a.–d. In a plane, the three angles in a triangle must add up to 180°. Of the sets of angles in this problem, 31b, 31c, and 31d have that sum.

e. A triangle cannot have two 90° angles because the sum of the measures of those angles would be 180°, leaving no degree measure left for the third angle.

Problem 32 *(Student page 37)* This is the "inverse" of Problem 18b. The picture shows what additions (dotted lines) need to be made to the original square (shaded). The light diagonal lines within the square help to show why the additions exactly double the square's area.

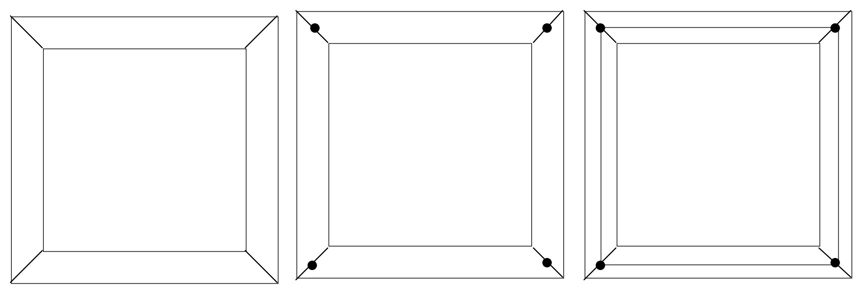


Problem 33 *(Student page 38)* Kings must generally be obeyed, but they are not always right! If each edge of a cube is doubled, the volume is multiplied by eight. The volume of a cube is the cube (!) of the side length. So, if a cube's side is a, its volume is a^3. A cube whose side is $2a$ has a volume of $(2a)^3 = 8a^3$.

Problem 34 *(Student page 38)* If each edge of a new cube matches the diagonal of a face of the original cube, the *area of each face* of the new cube is twice the area of each face of the original. This is the result you obtained in Problem 32. But the volume of the new cube is almost tripled.

Problem 35 *(Student page 38)* A cube that has side length 1 inch has a volume of 1 cubic inch. A cube with double that volume (2 cubic inches) must have a side length of $\sqrt[3]{2}$ inches. More generally, to multiply the volume of a cube by n, the sidelength must be multiplied by the cube root of n.

Problem 36 *(Student page 40)* To construct a square "halfway between," connect the corresponding vertices of the two squares, find midpoints of these connecting segments, and then connect those midpoints in order.

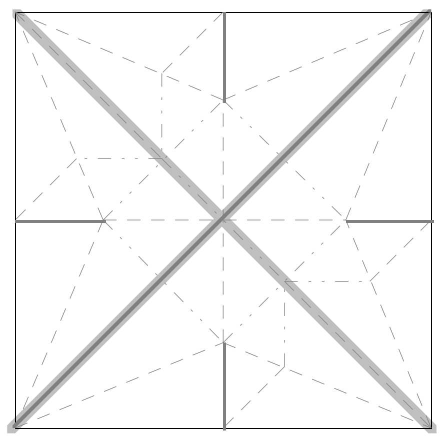


Problems 37–38 *(Student page 40)* A square circumscribed around a circle has an area about 27% larger than the circle. (Its area is d^2 instead of $\pi(\frac{d}{2})^2$.) The square inscribed in the circle has only about 64% the area of the circle. (Rotating it 45° shows the same configuration you've seen in Problem 18b and again in Problem 32. Its area is $\frac{d^2}{2}$.) A square whose sidelength is exactly midway between those of the two squares is quite close to the area of the circle: the area of the circle is only about 8% larger. If halfway from smaller to larger is not large enough, perhaps three fifths of the way (or some other fraction) would work? Of course, there certainly *is* some point at which the square has the exact same area as the circle! But *finding* that point—the right position for the vertex, or the right length for the side—can only be done by approximation, and not by construction (with compass and straightedge).

CONSTRUCTING FROM FEATURES: PAPER FOLDING

Problem 2 *(Student page 46)* Any angle other than 45° will not make the three corner planes perpendicular to each other.

Problem 3 *(Student page 55)* The following crease pattern includes only the creases made through Step 12. Three different styles of lines are used. The dashed pattern indicates valley folds. The dash-dot-dot pattern indicates mountain folds, and the gray lines indicate "construction" creases—places where creases had originally been made, but which lie completely flat (unfolded) in the finished bird.



Problem 4 *(Student page 55)* The two diagonals are lines of symmetry.

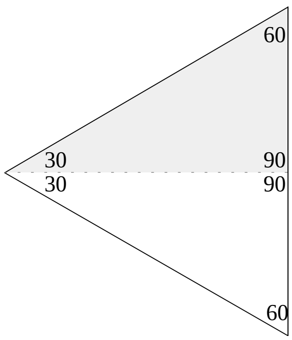
Problem 5 *(Student page 55)* This is a very challenging task. There are several ways to fold the required square. The solution that uses the fewest folds may not be the one that uses the least geometric knowledge! We will show one solution.

The Pythagorean Theorem—$a^2 + b^2 = c^2$—suggests a couple of easy options. If a^2, b^2, and c^2, were 1, 2, and 3, respectively, then a, b, and c would be 1 (which is given as the original side length), $\sqrt{2}$ (the length of the diagonal of that square), and $\sqrt{3}$ (twice the side of the square we're looking for), respectively. Also, if a^2, b^2, and c^2, were 1, 3, and 4, respectively, then a, b, and c would be 1, $\sqrt{3}$ (again twice the side of the needed square), and 2, respectively.

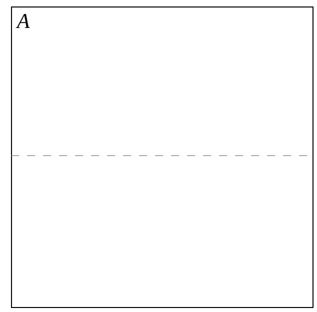
At first, there appears to be a problem with the second one. Though it is easy enough to *draw* a segment of length 2, we cannot *create* the length 2 by folding a square whose sides are 1: there isn't enough paper! Still, the length that we *really* want is $\frac{\sqrt{3}}{2}$, which we could get by cutting all the sides in half: $\frac{1}{2}$, $\frac{\sqrt{3}}{2}$, and 1.

CONSTRUCTING FROM FEATURES: PAPER FOLDING (continued)

A right triangle whose shorter leg is half the hypotenuse is a 30°–60°–90° triangle.

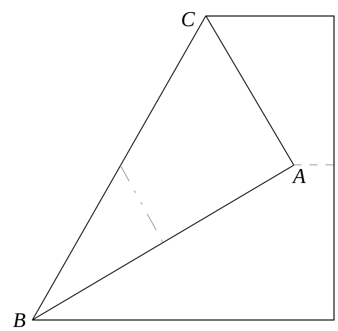


The solution we will show takes five folds, and is worth understanding as an easy way to create equilateral triangles by paper folding. Can you find the four-fold solution?



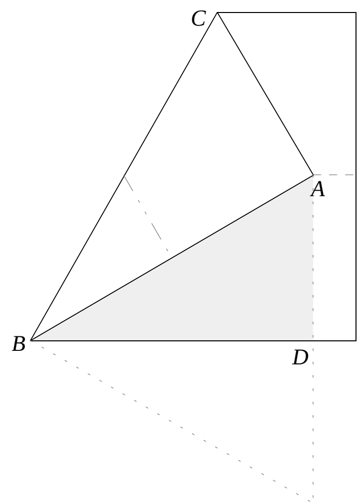
Step 1: Create the length $\frac{1}{2}$ by folding the square in half and unfolding again, as shown here.

Step 2: Bring corner A to the center crease in such a way that $\overline{AB}$ lies flat on the paper and the new crease passes through B. You have just created a 30°–60°–90° triangle: $\triangle ABC$.



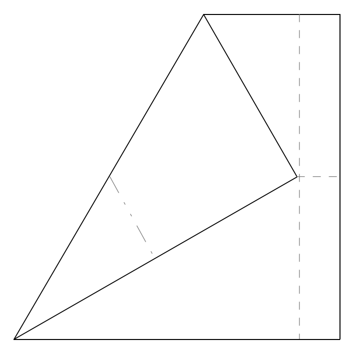
Why does this work? It is probably easiest to see why by looking at $\triangle ABD$. The length of $\overline{AB}$ is 1, the length of $\overline{AD}$ is $\frac{1}{2}$, and $\overline{BD} \perp \overline{AD}$. (A fact about parallel lines that comes up later in the Student Module also shows that the original fold that divided

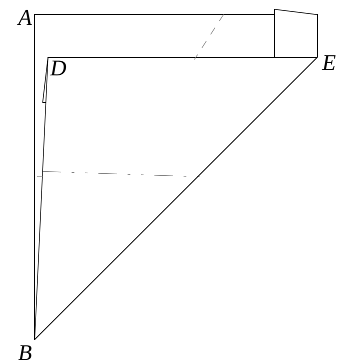
the square in half also divides $\overline{BC}$ in half and guarantees that $\overline{AC}$ is half the length of $\overline{BC}$.)



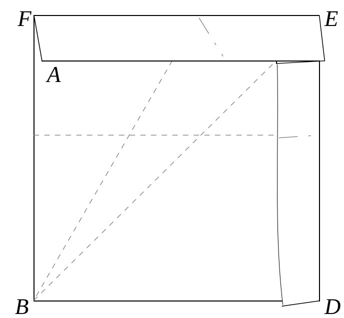
This means that $\overline{BD}$ is the required length: $\frac{\sqrt{3}}{2}$. The rest of the construction is straightforward.

Step 3: Fold away the excess, using the corner (A) as a guide.





Step 4: Transform this rectangular shape into a square in the usual way. "Copy" the length BD onto $\overline{AB}$ by folding one against the other through B. This marks the location E.

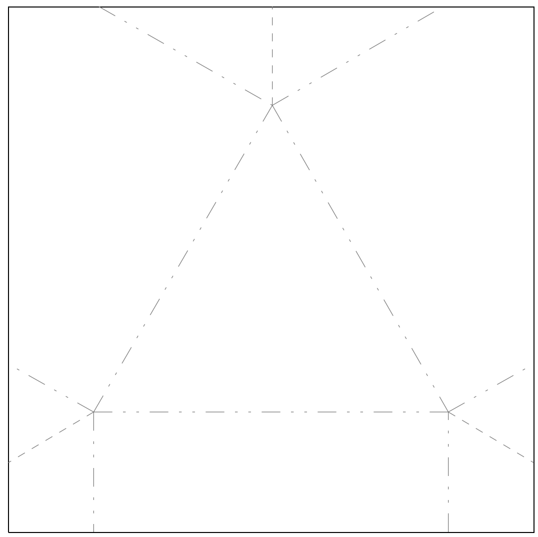


Step 5: Unfold the last fold, and fold the "excess" of the rectangle down at E. The result, square $BDEF$, is $\frac{3}{4}$ the area of the original square.

Problem 7 *(Student page 57)* This not only folds flat, but has a kind of spring-like nature that keeps it flat.

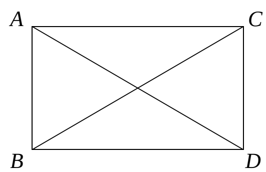
CONSTRUCTING FROM FEATURES: PAPER FOLDING *(continued)*

Problem 8 *(Student page 58)* One solution is shown below. The additional creases are angle bisectors.

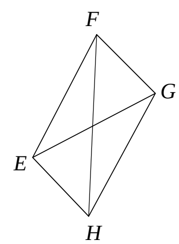


CONSTRUCTING FROM FEATURES: GROUP THINKING

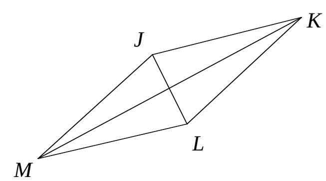
Puzzle 1. For six segments (Clue 4) to divide the plane into five regions (Clue 3), four of the segments might create a quadrilateral with two parallel sides (Clue 4), while the remaining two segments divide that quadrilateral into four regions. Those remaining segments must bisect each other (Clue 2), and could be diagonals. There are at least three ways for this to be done using line segments of three different lengths (Clue 1).



Method 1: ABCD is a rectangle.

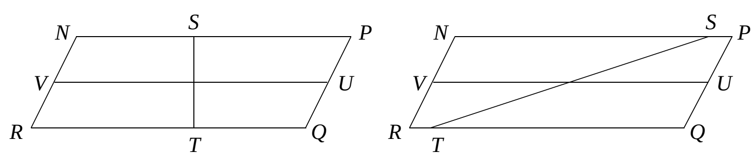


Method 2: $EF = EG = GH$



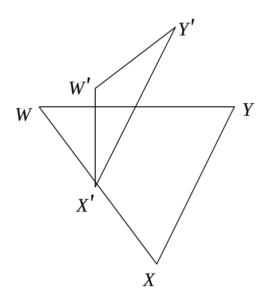
Method 3: $JK = KL = LM = MJ$

The two segments that bisect each other need not be diagonals.



Methods 4 and 5: $NP = VU = RQ$, $NR = PQ$

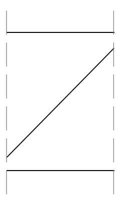
You can also solve the puzzle with two isosceles triangles, if $\overline{XY} \parallel \overline{X'Y'}$, $WX = WY$, $X'Y' = XY$; $\overline{X'Y'}$ and $\overline{WY}$ bisect each other.



Method 6: Two intersecting triangles

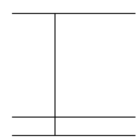
Without the optional clues, there are many different solutions. The optional clues narrow down the possibilities. Optional Clue 1 rules out Methods 2–5 by requiring there to be two segments of each of the three sizes. Method 1 will still work if the diagonals make 60° angles with the shorter sides, because they will then be twice the length of those sides. Method 6 will work if $WX = 2W'X'$ (as drawn). Neither Optional Clue 2 nor Optional Clue 3 narrow the solutions any further: It is possible to construct Method 6 so that $\overline{W'X'} \perp \overline{WY}$, satisfying Optional Clue 3, while $\overline{W'X'}$ crosses $\overline{WY}$ without being perpendicular to it, satisfying Optional Clue 2.

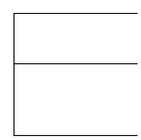
This is *not* a solution because the three segments do not intersect.



Puzzle 2 creates an upper case letter Z. Clue 1 says that there are three segments, and Clue 4 says that two of them are horizontal (and therefore parallel to each other), so Clue 3 and the first part of Clue 5 are redundant. The picture in the margin satisfies Clue 5, but Clue 6 requires the three segments to connect.

Puzzle 3. Two of many possible solutions are shown below. Clues 1–3 together state that this figure consists of four segments, three horizontal, and one perpendicular to those three.

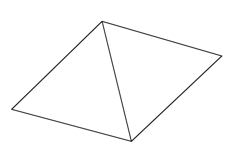


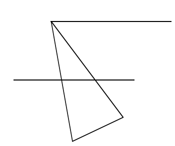


Are there other solutions?

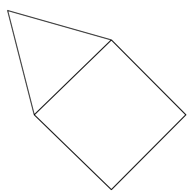
Puzzle 4 creates a line segment 4″ long, with two 2″ line segments, end to end, directly on top of it. Clues 1 and 6, together, imply that there are at least three line segments: the total length is eight inches, and there is only one segment as long as 4″. If there are only three segments, as Clue 5 implies, then Clues 1 and 6 further show that the two shorter segments must each be 2″ long (because neither can be longer than 2″), consistent with Clue 2. If those two short segments lie end-to-end on the longer segment, the resulting figure conforms with Clues 3 and 4.

Puzzle 5 creates a rhombus with 60° and 120° angles and with one diagonal dividing it into two equilateral triangles. Five segments (Clue 2), three of which are the same length and meet at a single vertex (Clue 4), and two of which are parallel (Clue 5), can divide the plane into three regions (Clue 3) in more than one way.



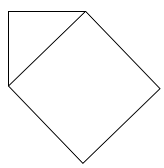


But Clue 6 requires the two inside regions to be the same size and shape, and Clue 1 requires that the five segments (with a total of ten endpoints) be arranged so that the resulting figure has only four vertices.



Puzzle 6 creates a "house" with a 60° roof. Is this the only solution?

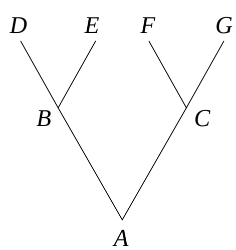
Puzzle 7 creates a "house" with a 45° roof. Clue 6 sets the stage: the drawing is a square with two segments attached to it. Clue 4 says that these two additional segments must create one more right angle. That can be done in more than one way, but Clues 3 and 5 show that the added segments must connect in ways that allow *other* segments to be removed and the result still be a polygon.



Puzzle 8. A little experimenting with Clues 1 and 2, by themselves, will show that the figure is either a rectangle with diagonals or a regular pentagon with diagonals. (Polygons with three sides have no diagonals; polygons with more than five sides have diagonals of more than a single length.) Clue 4 rules out rectangles except for a square, but it is redundant because any one of Clues 3, 5 and 6 is sufficient to rule out rectangles altogether.

Puzzle 9. The same reasoning that solves Puzzle 8 shows that this puzzle creates a (nonsquare) rectangle with diagonals.

Puzzle 10 creates a V with smaller Vs at the end of each "stem."



Solutions
Investigation **1.8**
Student Pages 61–63

Algorithmic Thinking: Directions for People

Problem 1 *(Student page 62)* When you compare maps, it would be a surprise if *any* segments or angles actually match, but there should be schematic similarities—right or left turns will follow each other in the same sequence on each map.

The directions given in the Student Module are *not* an algorithm for drawing the map. How large each block is, and what is meant by "a few blocks" is up to each mapmaker. An algorithm is a fully-specified procedure: any two people following it will produce exactly the same result.

Problem 2 *(Student page 62)* This tourist map does not show a parking lot, so we must assume that the directions were written by someone who knew the city itself, and wasn't describing only what can be seen on the map. According to the map, these directions take us from the Pike Place Public Market to The Space Needle.

If this map is to be trusted, the route we suggested is much shorter than the original, but still not *quite* the best. Can you find a route that is even shorter (if only by a tiny bit) and *prove* that it is shorter?

Problem 3 *(Student page 62)* Instead of taking 1st Avenue all the way to Denny Way, it would clearly be better (the Triangle Inequality again!) to turn right as soon as you reach Broad Street.

Problem 6 *(Student page 63)* Maps will differ. Wherever you must use knowledge not supplied in the written description, the written description is not a complete algorithm.

Solutions

Investigation **1.9**

Student Pages 64–68

ALGORITHMIC THINKING: DIRECTIONS FOR ROBOTS

Because this problem does not state which angle is meant, either answer is OK.

Problem 2 *(Student page 65)* There are at least two ways to think about angles. The angle between Denny Way and 1st Avenue is 45°, as one *sees* it on the map, but 135° (a lot of *turning*) to the turtle or to a driver.

Problem 3 *(Student page 67)* Here is one of many ways to write such a procedure:

```
To Block
 forward 40        two pairs of congruent sides
 right 90          four right angles
 forward 60
 right 90
 forward 40
 right 90
 forward 60
 right 90
end
```

Answers:
a. Picture4
b. Picture2
c. Picture1
d. Picture3

Problem 4 *(Student page 67)* Each picture contains two long segments and a short one. The order in which they are drawn helps you know which procedures drew them. The two procedures that have `forward 20` at the end can be distinguished by the *turns* they use. This part is the trickiest. Just as one *sees* a 45° angle on the map between Denny Way and 1st Avenue but must *turn* 135° when driving, the turtle's turns and the angles we see are different.

Problem 6 *(Student page 68)* One would need a ruler to make correct `fd` distances, and a protractor to measure correct turning angles `rt` (and something to draw with).

Vocabulary: pitch, yaw, roll

Problem 7 *(Student page 68)* A fish swimming in water may "pitch" toward its top side or toward its bottom side by any angle; it may "roll" counterclockwise or clockwise on its swimming axis (which doesn't change its swimming direction); and it may veer, or "yaw" toward its left or right fin. Even though rolling doesn't affect direction itself, it *does* alter how pitching and yawing will affect direction. If a fish has already rolled in such a way that it is swimming upside down, then a pitch toward its dorsal (top) side will move it *deeper* in the water!

Problem 8 *(Student page 68)*

a. "Not enough inputs to `forward`" or some similar message means that the command `forward` was not told how *far* forward to move the turtle.

b. The same kind of message is given for `rt` or `lt`, if the amount of turning is not specified.

c. Logo may report `you don't say what to do with 90` or `I don't know what to do with 90` if you don't specify whether one is to go *forward* 90 steps, or turn *left* 90 degrees, or just *print* 90 on the screen.

Logo may say something like `I don't know how to fd100` if you misspell a command, or forget a space between the command and its input.

ALGORITHMIC THINKING: ANGLES AROUND A CENTER

Problem 1 *(Student page 69)*

A procedure for the 8-spine star may look like this:

```
To 8star
 fd 25
 bk 25
 rt 45
 fd 25
  ⋮
end
```

a. If you are careful to check that the turtle really *is* back where it started at the end of each design, then the angles you find will be accurate enough to develop a good conjecture in Problem 2b.

b. Here's one way of checking: If all the `forward`s and `back`s are the same, the only question is whether the turtle ends up *facing* the same direction it started. That can be checked by eye if one simply does one more `forward 25` `back 25` at the end, and sees if the last line draws directly over the first. You could also use Logo's `heading` and `pos`ition operations by typing the commands `show heading` or `show pos`.

Problem 2 *(Student page 70)* Each figure divides up the "pie" into a different number of pieces, whose total central angle is 360°. So, in the three-part pie, the angle between the slices is 120°, for the 4-spine the angle is 90°, and so on.

A generalization about the necessary turning angle:

$$\text{TurnAngle} = \frac{360}{\text{\# of Spines}}$$

Number of Spines	Amount of Turning between Spines
3	120
4	90
5	72
6	60
8	45
9	40

Problem 3 *(Student page 71)* In Algorithm 1 there is a `fd 35` instead of `fd 25`. In Algorithm 2 there is a `rt 130` instead of `rt 120`. Most people find it easier to notice the error in the second algorithm because there are fewer commands and numbers.

Problem 4 *(Student page 71)* One approach:

```
spine
rt 45
spine
rt 45
      ⋮
spine
rt 45
```

Problem 5 *(Student page 72)* Here is how the 3-spine star might be made, using `repeat`.

```
repeat 3 [spine   rt 120]
```

The others are done similarly, changing only the number after the repeat and the angle.

Problem 6 *(Student page 72)* Converting these algorithms into named procedures requires only the addition of a title line and "end." For example:

```
to 3point                        or any other title
 repeat 3 [spine   rt 120]
end
```

Problems 7–8 *(Student pages 72–73)* The two solutions shown below are the most straightforward.

To Design1	To Design2
fd 60	fd 20
rt 90	rt 120
fd 20	fd 20
rt 90	rt −60
fd 10	fd 20
rt −90	rt 120
fd 40	fd 20
rt −90	rt −120
fd 10	fd 20
rt 90	rt 120
fd 20	fd 20
rt 90	rt 60
fd 60	fd 20
rt 90	rt 60
fd 20	fd 20
rt −90	rt −120
fd 10	fd 20
rt 90	rt 120
fd 40	fd 20
rt 90	rt 60
fd 10	fd 20
rt −90	end
fd 20	
rt 90	
end	

The solution on the left takes 24 commands: 12 sides (fd), and 12 turns (rt). Eight of the turns are 90°, and 4 are −90°: $8 \times 90 + 4 \times (-90) = 360$.

The solution on the right takes 21 commands: while there are only 10 sides, there are 11 fds because the turtle starts in the middle of a side. Summing the 10 turns, we get: $4 \times 120 + 2 \times (-120) + 3 \times 60 + 1 \times (-60) = 360$.

Problem 9 *(Student page 73)* There are several ways to write an algorithm to draw a rectangle twice as long as it is wide.

You may write it simply using fd and rt and a particular choice of numbers, like this:

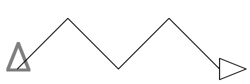
```
fd 60
rt 90
fd 30
rt 90
fd 60
 ⋮
```

You may use repeat to show the structure more clearly, like this:

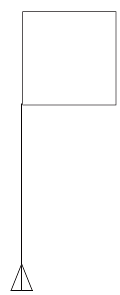
```
repeat 2 [ fd 60  rt 90  fd 30  rt 90 ]
```

And you may package either of these forms into a procedure by adding a title line and end.

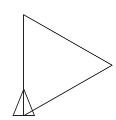
Problems 10–11 *(Student pages 73–74)*



Mystery Fig 1



Mystery Fig 2



Mystery Fig 3

Problem 12 *(Student page 74)* back −25 acts like forward 25. It is essentially a double negation: $-(-a)$.

Problem 14 *(Student page 75)* Here's one way to create a seven-spined star:

```
to 7point
 repeat 7 [spine   rt 360/7]
end
```

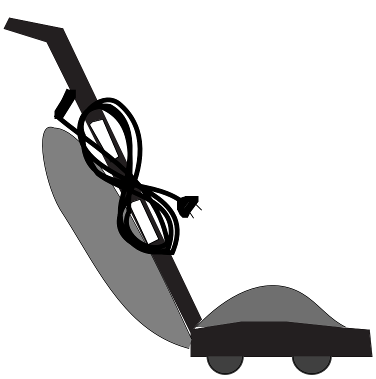
Problem 15 *(Student page 75)* Forward or backward motion does not increase or decrease the amount of turning, so imagine drawing these figures with such small forward distances that the turtle seems to rotate in one spot. (Or imagine watching from very far away and still somehow seeing how the turtle is *facing* even though all the distances seem to vanish.) In the first figure (the quadrilateral) the four clockwise turns rotate the turtle through a full turn, 360°. In the second figure, it is impossible to distinguish separate turns, but if one attempts to face one's hand in the direction the turtle must face (or to walk the path), one sees, again, a full rotation. The third figure does some "unturning" along the way, but makes up for it, again turning the turtle through a full turn of 360°.

Problem 16 *(Student page 75)* At this stage, all that the experiments seem to show might be summed up this way:

ATTEMPTED THEOREM ***Total Turtle Turning Theorem (preliminary version)***

If the turtle takes a trip and returns precisely to where it started—same position and same heading—it will have turned a total of 360°.

Despite the difficulty of finding a clear and correct answer for Problem 17, the problem contains a practical piece of geometry you can use at home to preserve the life and safety of electrical cords! When rolling up extension cords or the power cord for a vacuum cleaner, it is best not to twist them. Looping them around in a circle introduces a lot of twisting (one twist for each time around the loop). Looping the cords in a figure eight introduces no twists. Try it to see.



Problem 17 *(Student page 76)*

a. In the figure-eight-like picture (and in its polygonal variant), the total turning is 0°; in the loops, the total turning is 720°. The simplest revision might be this (though it's a bit hasty to make any conclusion on the basis of only two experiments and no analysis):

THEOREM ***Total Turtle Turning Theorem (version 2)***

If the turtle takes a trip and returns precisely to where it started—same position and same heading—its total turning will be a multiple of 360°.

A slightly sharper statement (again, merely on the basis of observation) might be:

THEOREM *Total Turtle Turning Theorem (version 3)*

If the turtle takes a trip and returns precisely to where it started—same position and same heading—its total turning will be a multiple of 360°. If the turtle's path never crosses itself, the total turning will be exactly 360°.

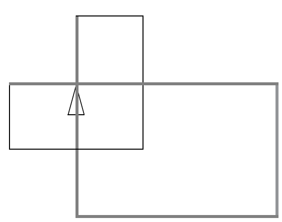
In fact, these *are* true statements.

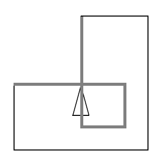
b. The total turning in the five-pointed star is 720°. It can be drawn with Logo this way: `repeat 5 [fd 50 rt 720/5]`. The 10-point star rotates the turtle three full times before returning it to its starting position: `repeat 10 [fd 50 rt 3 * 360/10]`. The third path rotates the turtle only one full turn through 360°. On the basis of the stars, it is tempting to conjecture the following:

ATTEMPTED THEOREM *Total Turtle Turning Theorem (addendum 1)*

If all turns are positive and in one direction (all clockwise or all counter-clockwise), then if the turtle crosses its path n times, the total turning will be $360(n + 1)$.

Unfortunately, a simplified version of the third picture shows that this is not quite right. One of the paths below contains three self-crossings. In the other, three of the segments are shorter, resulting in only one self-crossing, but none of the turns are changed at all, so the total turning must be the same.





It is *extremely* tricky to make a clear statement about what happens if the path includes mixed (clockwise and counterclockwise) turning or multiple self-crossings.

Algorithmic Thinking: Spines, Stars, and Polygons

Problem 2 *(Student page 77)* VarSpine needs more inputs. It means that VarSpine, like fd and rt, needs a number (which represents the length of the spine) to go along with the command.

Problem 4 *(Student page 78)* Here is one way to write it:

```
To BetterStar :NumSp
 repeat :NumSp [VarSpine 25  rt 360/:NumSp]
end
```

The command BetterStar 5 will draw a five-pointed star.

Problem 5 *(Student page 78)*

a. The Shape1 algorithm moves the turtle forward and then back to its starting position before turning. The Shape2 algorithm moves the turtle forward, turns, and continues from there. As a result, Shape1 will draw an *n*-spined star, while Shape2 will draw an *n*-sided regular polygon.

b. Shape1 5 is a 5-spined star, Shape2 5 is a pentagon, Shape1 10 is a 10-spined star, and Shape2 10 is a decagon.

Problem 7 *(Student page 79)* The exploded star is a hybrid of Shape1 and Shape2. To create the exploded star, the turtle goes forward and comes back part way: in Shape1 it comes back all the way; in Shape2 it doesn't come back at all. In all these cases, the turtle turns the same angle.

Problem 8 *(Student page 80)* In the Shape algorithm, the turtle is using exterior angles.

Problem 9 *(Student page 80)* The exterior angles of a pentagon measure 72°. It is the same as the interior angle on the 5-spined figure, which is $360^\circ/5 = 72^\circ$.

Problem 10 *(Student page 80)* A "straight angle" is 180°, so the interior angle is $180^\circ - 72^\circ = 108^\circ$. All the exterior angles are the same size (because the procedure specifies repeated turns of 72°), so all interior angles must be the same size.

Problem 11 *(Student page 80)* Between each pair of sides is a turn, so 36 sides means 36 turns. Because the total turning is 360°, each turn (exterior angle) is 10°. Therefore, each interior angle is 170°.

Problem 12 *(Student page 80)* One group of students wrote: "We divided 360 by the number of sides of the polygon we wanted to make, and that gave us the exterior angle. Then we subtracted that from 180 to get the interior angle." Here is the same idea translated into the language of algebra:

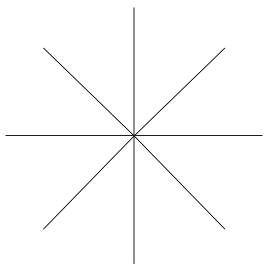
$$\textit{interior angle} = 180 - \frac{360}{n}, \text{ where } n = \text{number of sides.}$$

Problem 13 *(Student page 80)*

- The `fd` and `bk` commands require one numerical input. They tell the turtle to move forward or backward that distance in "turtle steps." If the turtle's pen is down, the turtle draws a line of that length.
- `rt` and `lt` require one numerical input, and tell the turtle to rotate right (clockwise) or left (counterclockwise) from the direction it is currently facing.
- The term `to` is used for defining a new process. It introduces a procedure name, and tells Logo that the set of commands that follow are not to be executed immediately, but are to be bound together as a new process that will be run later.
- `cs` clears the drawing screen. It takes no inputs.
- `repeat` takes two inputs: a positive whole *number*, and a list containing one or more *instructions*. It executes the instructions as many times as indicated by the number.
- `PikeToNeedle` draws one possible path (not to scale) from Pike Place Market to the Space Needle in Seattle. It takes no inputs.
- `Spine` draws a 25 turtle-unit line and returns the turtle to the position it was in before drawing that line. It takes no inputs.
- `VarSpine` requires one numerical input, draws a line of that length (in turtle units), and is *state-transparent* (that is, returns the turtle to the exact position and heading it was in before drawing that line).
- `8point`, `5point`, and so on are state-transparent commands that take no inputs and draw "spiny stars." `8point` draws an eight-pointed "star"—eight 25-step segments radiating out from a central point and 45° apart; `5point` draws the same sort of star, but with five segments 72° apart; and so on.
- `Star` is similar to `8point`, but its inputs allow it to draw stars of any number of points specified. It requires two numerical inputs: a number of points, and a turning angle (in degrees). It draws the specified number of line segments radiating out from a central point separated by the specified angle. If the product

When a set of movements returns the turtle to the position it was in before starting the movements, we call that set of movements "state-transparent."

of the number of segments and the turning angle is a multiple of 360, then this command draws pictures like `8point` and is state-transparent. If the product of the two inputs is not 360, the command draws irregular "stars."



Problem 14 *(Student page 81)* `shape1 4 rt 45 shape1 4` draws the same picture as `Shape1 8`: an 8-spined figure with spines of length 25 and 45° angles between adjacent spines.

Problem 15 *(Student page 81)* There are many ways to do each figure.

For Figure 1, we first edit `Shape2` to make a slightly bigger polygon:

```
To Shape2a :num
 repeat :num [fd 45  rt  360/:num]
end
```

To get the length for the side of the polygon, we simply played around in Logo to figure out what worked. To get the turning angle of 150°, we realized that the spines would divide each 60° angle of the triangle in half, and we subtracted 180° – 30°.

We can then use this and `BetterStar` to create the figure:

```
betterstar 3
fd 25 rt 150
shape2a 3
```

For Figure 2:

```
repeat 4 [shape2 4  rt 90]
```

For Figure 3:

```
repeat 6 [shape2 3  rt 60]
```

Problem 16 *(Student page 81)* There are many ways to do these. For both, it is a big help to have polygon- and star-drawing procedures that allow for variable size.

```
To VStar :num :length
 repeat :num [VarSpine :length   rt 360/:num]
end

To VPoly :num :sidelength
 repeat :num [fd :sidelength   rt 360/:num]
end
```

The central part of the hexagonal web can be seen as nested hexagons, or as six sets of nested equilateral triangles. Here is one way of drawing it as a set of nested hexagons:

```
To HexaWeb
 vstar 6 80
 fd 10
 rt 120  vpoly 6 10  lt 120
 fd 10
 rt 120  vpoly 6 20  lt 120
 fd 10
 rt 120  vpoly 6 30  lt 120
 fd 10
 rt 120  vpoly 6 40  lt 120
end
```

The pentagonal web is harder. One can plan it in essentially the same way as the hexaweb: start with a `vstar 5 80`, and then, from the center, move out to successively larger and larger shells. For each shell, turn in the appropriate direction and draw a pentagon of the appropriate size.

Each fifth of the web is a nest of triangles with a common vertex at the center of the five-spined star. The measure of the central angle (the angle at the common vertex) is $\frac{360^\circ}{5} = 72^\circ$, so the remaining two angles in each triangle must each be 54° (because $54 + 54 + 72 = 180$). The turtle, therefore, must turn $(180 - 54)^\circ = 126^\circ$. To get the lengths requires some trigonometry, but you can make good approximations with trial and error. Here is one procedure that just about works:

The `home` command takes the turtle back to the original heading and position. Rather than continuing from where he finished one pentagon, he returns home before starting the next one. This prevents small errors from building up into larger errors. This same program without the `home` commands does not do as nice a job.

```
To PentaWeb
 vstar 5 80
 fd 10
 rt 126  vpoly 5 12  lt 126
 home fd 20
 rt 126  vpoly 5 23  lt 126
 home fd 30
 rt 126  vpoly 5 34  lt 126
 home fd 40
 rt 126  vpoly 5 45  lt 126
end
```

Problem 17 *(Student page 82)*

```
to Shape3 :num
 repeat :num [fd 100/:num  rt 360/:num]
end
```

As the number of sides grows, the sidelength decreases and the size of the interior angles of the polygon increases. In fact, as the number of sides increases, the shape begins to resemble a circle.

If you draw `Shape3 3` and `Shape3 4` without clearing the screen in between, you might be able to see that the square contains more area than the triangle. In fact, each successively greater number of sides increases the area, but there is a limit to how big that area can get!

The *Connected Geometry* module *Optimization* investigates the problem of "isoperimetric" (same perimeter) figures in depth.

The following are true statements, but not all are observable on the basis of this experiment.

- Among polygons with a given perimeter and a given number of sides, the *regular* polygon has the greatest area.
- Of regular polygons with a given perimeter, the one with the most sides has the greatest area.
- Of all figures with a given measure around their boundary, the circle has the greatest area.

Problem 18 *(Student page 82)* A wide range of observations is possible. Here is one: if you use 720 instead of 360, then commands that produced polygons with $2n$ sides now produce polygons with only n sides.

ALGORITHMIC THINKING: IRREGULAR FIGURES

The word "trigonometry" is made of two parts: *trigon* (triangle) and *metry* (measuring). Trigonometry lets you solve problems like this *without* experiments.

Problem 1 *(Student page 83)* Without trigonometry, this problem can be solved only by experimentation. Starting with a variable angle and a "ray" can help. For example:

```
To TriangleBlock :turn
 fd 100
 lt 90
 fd 200
 lt :turn                  We'll experiment with this.
 fd 400                    a "ray" long enough to let me experiment
                           with the angle
 lt 360 – 90 - :turn       (Why is this computation used?)
end
```

A little experimenting shows the best results with a turn of approximately 153.5° (the command TriangleBlock 153.5). Without the Pythagorean Theorem, the length of the hypotenuse can also be approximately determined by experimentation. The procedure might be changed as follows:

```
To TriangleBlock :hypotenuse
 fd 100
 lt 90
 fd 200
 lt 153.5                  discovered by experimenting
 fd :hypotenuse            Now we'll experiment with this.
 lt 360 – 90 – 153.5
end
```

Another way to complete the procedure is by using the home command introduced in the solution to Problem 16 in Investigation 1.11. This will bring the turtle back exactly to the position and heading from which it started, drawing the segment necessary to get there. Try it!

By the Pythagorean Theorem, the required distance is about 223.607. Experiments show that distance to be about 224 steps. This is close enough for a drawing at this scale!

Problem 3 *(Student page 83)* The theoretical total turning (total exterior angles) is 360°. The procedures used for experimentation that were described above had that total built into them (360 – 90 – 153.5). If you determined each non-90° angle experimentally, your total may not be precisely 360°.

Problem 4 *(Student page 84)* You have already reasoned, in several different problems, that the total turning in a closed figure (that does not cross itself) is 360°.

Problem 5 *(Student page 84)* You may use the fact that a straight angle is 180° and

do the computations, or you may measure the angles using a protractor. Computations based on the experimentally-derived value of 153.5 would produce a table like the following:

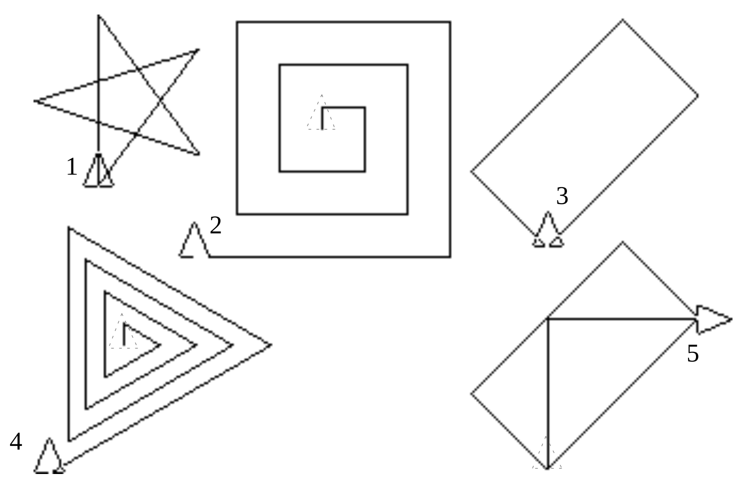
Turn Command	Exterior Angle	Interior Angle
lt 90	90	90
lt 153.5	153.5	26.5
lt 116.5	116.5	63.5

Problem 6 *(Student page 84)* If the measures of the three exterior angles of a triangle are a, b, and c (whose sum is 360°, because the sum of the exterior angles of any polygon is 360°), then the measures of the three interior angles are $180 - a$, $180 - b$, and $180 - c$. The sum of the measures of those angles is

$$(180 - a) + (180 - b) + (180 - c) = (180 + 180 + 180) - (a + b + c) = 180.$$

Therefore, the sum of the measures of the interior angles of any triangle is 180°.

Problem 7 *(Student page 84)* The designs are numbered in the picture below to correspond to the programs that will create them.




```
to Design1
 repeat 5 [fd 50  rt  720/5]
end
```

ALGORITHMIC THINKING: IRREGULAR FIGURES
(continued)

Can you think of a clever way to rewrite Design2 using just one line and the repeat command?

```
to Design2
 fd 10 rt 90
 fd 20 rt 90
 fd 30 rt 90
 fd 40 rt 90
 fd 50 rt 90
 fd 60 rt 90
 fd 70 rt 90
 fd 80 rt 90
 fd 90 rt 90
 fd 100 rt 90
 fd 110 rt 90
 fd 120 rt 90
end
```

```
to Design3
 lt 45
 repeat 2 [fd 30 rt 90 fd 60 rt 90]
end
```

Can you think of a clever way to rewrite Design4 using just one line and the repeat command?

```
to Design4
 fd 10 rt 120
 fd 20 rt 120
 fd 30 rt 120
 fd 40 rt 120
 fd 50 rt 120
 fd 60 rt 120
 fd 70 rt 120
 fd 80 rt 120
 fd 90 rt 120
 fd 100 rt 120
 fd 110 rt 120
 fd 120 rt 120
end
```

```
to Design5
 lt 45
 repeat 2 [fd 30 rt 90 fd 60 rt 90]
 rt 45
 fd 42.5
 rt 90
 fd 42.5
end
```

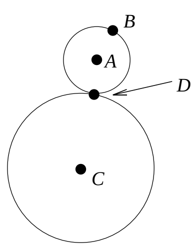
CONSTRUCTING FROM FEATURES: MOVING PICTURES

Problem 1 *(Student page 86)*

a. You can make a triangle out of line segments using only the segment-drawing tool, or you can place the three points first and then connect them with segments. You can move the vertices of the resulting triangle. The triangle will change shape accordingly, but will remain a triangle. What happens when you move the vertices so that they are collinear (lying along a single line)?

b. Make sure that the segment's endpoints are *on*, rather than *in* (or merely near) the circles. How are the different parts of the figure constrained when you try to move them?

d.–e. Rays and lines can be drawn with the same tool that draws segments. They can also be constructed in other ways.



Problem 2 *(Student page 86)* Here is one way to draw the picture: Use point A as the center of the small circle and point B to anchor the circle. As you build your circle, make sure your pointer goes directly to point B. If you do not, the circle may appear to pass through B, but subsequent draggings will be able to change that.

Then create two new points, one (C) considerably below the small circle and one (D) on it, roughly at its bottom. Draw the second circle with C as its center and D as a point on its circumference. Again, be sure to *use* point D as the anchor for the circumference. Dragging the center of either circle can change its size and position; dragging B makes changes in *both* circles. (Why?) When D is dragged, only the circle about C changes.

Problem 3 *(Student page 87)* To constrain one vertex of the triangle to move only around a fixed circle, create a circle (in any way) *first*, then place a *new* point on it, and then *use* that point in constructing the triangle. To get a picture *exactly* like the one shown in the Student Module, you'll need to hide some points.

Problem 4 *(Student page 87)* To constrain the vertices of the triangle sufficiently, construct the lines first. The vertices of the triangle must be placed on already-existing lines and must not be points that help to specify the lines.

Problem 5 *(Student page 89)* If all the objects are *constructed* according to their definitions, then the circle will move along with A as A is dragged; the size of the

circle and the slopes of the two lines through it will adjust to match the size and slope of segment $\overline{BC}$.

The term "slope" is often used to describe the direction of a line—in fact, it is the measurement that your software provides for that purpose—but the idea of "slope" requires a coordinate system on the plane. The fact that there is only one line through a given point at a given direction does not require any coordinate system. So, for our current purpose, we prefer the informal term "direction."

Problem 6 *(Student page 89)* In a plane, given a "direction" and a point, there is only one line that "travels in that direction" and passes through that point.

In constructing a perpendicular to a line, you use this idea. By selecting the line, you are determining the slope of the perpendicular. But there are infinitely many lines of that slope (infinitely many places to draw that perpendicular). By selecting the point, too, you are limiting the possibilities to just *one* perpendicular. (If you were to select *only* the point, then an infinite number of lines could pass through it—all of different slopes, and not just the desired perpendicular.) The same reasoning applies to the construction of parallel lines.

Problem 7 *(Student page 90)*

a. When you rotate $\overline{BC}$, the parts of the windmill should rotate as well; since the two lines were *constructed* perpendicular to and parallel to $\overline{BC}$, they must stay that way. The length of all four blades will remain equal to the length of $\overline{BC}$.

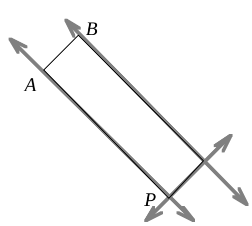
b. When you stretch or shrink $\overline{BC}$, the segments that make up the windmill will stretch and shrink to match. Since $\overline{BC}$ is defined as the radius of the circle, the (hidden) circle stretches and shrinks, and so do its radii (which form the windmill).

Problem 10 *(Student page 91)* There are many ways to construct a guaranteed rectangle. All good strategies start from essential features of rectangles. Here are some examples:

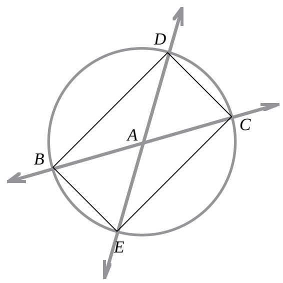
Properties of rectangles

- four right angles (four constructed perpendiculars)
- one pair of perpendicular sides (one right angle) and opposite sides parallel
- congruent bisecting diagonals (for example, two diameters of the same circle).

Each of these suggests a different construction. Here are two possible constructions:



A focus on right angles: Draw $\overline{AB}$ and construct perpendiculars at each endpoint (two guaranteed right angles). Place a point P on either of the lines and construct a perpendicular to that line at P. We've got it! Construct the needed segments and hide the infinite lines.

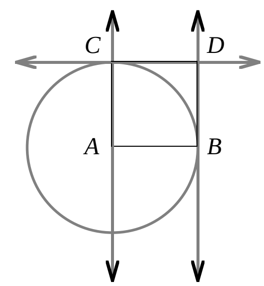


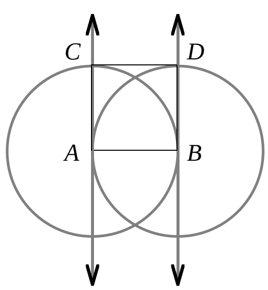
A focus on diagonals: Draw a circle with center at A. Place point B on the circle and draw a *line* through A and B; then place point C at the intersection of this line with the circle. In a similar way, construct points D and E as endpoints of a second diameter of the circle. These two diameters are the diagonals of the desired rectangle. Construct the sides of the rectangle and hide what isn't needed.

Problem 11 *(Student page 91)* Below are four common solutions. Still others are possible.

Properties of a square

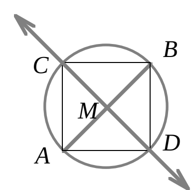
Two adjacent congruent sides, four right angles: Construct $\overline{AB}$, with perpendiculars at each end. A single circle with center A and radius $\overline{AB}$ locates the point C. A line through C perpendicular to $\overline{AC}$ (or parallel to $\overline{AB}$) completes the square.

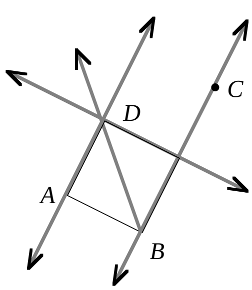




Three congruent sides with right angles between them: Construct $\overline{AB}$, with perpendiculars at each end. Two circles of radius AB and centers at A and B will copy the length AB along the perpendiculars, locating points C and D.

Perpendicular, congruent, bisecting diagonals: Construct $\overline{AB}$, its midpoint M, and its perpendicular bisector. Construct a circle centered at M with radius AM.





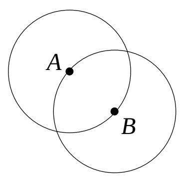
Imitating a paper-folding strategy: Construct $\overline{AB}$, with perpendiculars at each end of it, and place point C anywhere along one of the perpendiculars. Select points A, B, and C in that order to identify $\angle ABC$, and bisect $\angle ABC$ to locate D, a third corner of the square. An appropriate parallel (or perpendicular) will locate the last vertex.

Problem 12 *(Student page 91)* One solution uses this definition of a parallelogram:

DEFINITION

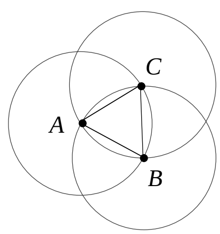
A ***parallelogram*** is a quadrilateral whose opposite sides are parallel.

Draw two arbitrary segments sharing a common endpoint to form two adjacent sides of the quadrilateral. Through the "free" endpoint of one segment, construct a line parallel to the other segment. Complete the construction.



Problem 13 *(Student page 92)* Draw a circle with center A. Place a single point B on that circle (not inside the region it bounds). Use B as the center of the second circle and use the center of the first circle to define the radius of the second circle.

Problem 14 *(Student page 92)* Use one of the intersection points of the circles in the previous figure as a center for a third circle. Use the center of either of the first two circles to define the radius of the third circle.

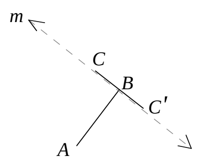


Construct circles around A and B as before. Use C, an intersection of the two circles, and A as center and radius point for a third circle.

Problem 15 *(Student page 92)* Two circles that pass through each other's centers (the solution for Problem 13) define all the necessary vertices for an equilateral triangle: the two centers and either of the two points where the circles intersect.

It is also possible to rotate a segment (*and* its endpoints) by 60° around one endpoint. The most common mistake with this strategy is to forget that the other endpoint must be part of the rotation to provide an attachment point for the final side of the triangle. Anyone using the rotation or translation menus for this construction should provide a thorough explanation of how it works.

Problem 16 *(Student page 92)* One way to center the stem of the "T" is to build it first, with a perpendicular at one endpoint, and then use a circle centered at the same endpoint to mark off two equal distances along the perpendicular.



Build m perpendicular to $\overline{AB}$. Place C on the perpendicular; C′ is a reflection of C over $\overline{AB}$.

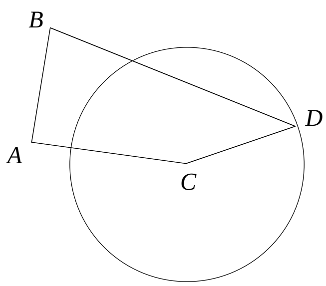
Alternatively, create the top first, and use its midpoint to locate the perpendicular line on which to build the stem.

Problem 17 *(Student page 93)* Here are two ways to think about the problem:

Rectangle first: Build a rectangle any way you like. Choose one corner, and define three rays, one through each of the other three corners. Place an arbitrary point on the "diagonal ray," and from it, construct lines parallel to the two sides of the original rectangle to create a second rectangle.

Diagonal first: Create two rays with a common endpoint—one to use as a diagonal, and the other as a base for the rectangles. Construct a perpendicular to the "base ray." Choose two points on the diagonal ray and, from each of them, drop perpendiculars to the line and the other ray.

If you construct this figure in any UnMessUpable way, it will guarantee that the sides of the two rectangles are proportional, meaning that the rectangles are geometrically "similar"—not merely both rectangular, but the *same* rectangular shape, differing only in size. The ratios $\frac{z}{y}$ and $\frac{u}{v}$ will change as the rectangles are made fatter or skinnier, but the two ratios will remain equal to each other, just as the two rectangles remain equally fat or skinny (the same shape, differing only in size).

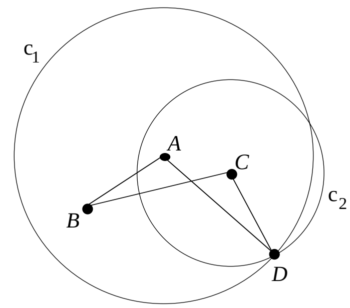


Segment $\overline{AB}$ defines the radius of the circle, and so $\overline{AB}$ and $\overline{CD}$ are guaranteed to be congruent.

Problem 18 *(Student page 93)* One method is to start with $\overline{AB}$, choose some point C and construct a circle of radius AB around C. Any point D on the circle will guarantee that $AB = CD$. Connecting the four points as shown does *most* of the job.

The sidenote to this problem asks for *both* pairs of opposite sides to be equal in length but not parallel. The same method can be used twice, but there is a choice to be made

at the end: the circles intersect in two places, and choosing the wrong one leaves the opposite sides parallel.



Circle c_1 centered at A has radius BC;
Circle c_2 centered at C has radius AB.
So $AB = CD$ and $AD = CB$.

We know of no way to make an UnMessUpable nonparallelogram with opposite sides equal in length, though! If you drag the vertices around, this butterfly figure occasionally turns into a parallelogram.

Problems 20–21 *(Student page 94)*

Midpoint, Perpendicular Bisector, Angle Bisector**:** The authors used these frequently in creating the pictures for the origami instructions in the Student Module.

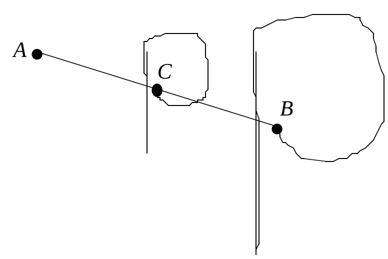
What is the difference between Angle and Slope?

Measurements (Distance**,** Area**,** Angle**,** Slope**):** You will use these extensively, along with **calculations** (for example, Ratio) performed upon them in experiments on geometric objects.

Polygon **or** Polygon Interior**:** You may need to construct these explicitly before you can measure the polygon's area or perimeter.

Rotate . . . **or** Reflect . . . **:** Sometimes, the most convenient way to think about a construction is to notice a symmetry. In those cases, rotation and reflection may be helpful tools.

Trace**:** Usually works like an on/off switch; any object or set of objects can be selected. The effect is not immediately visible, but when the objects are moved, they leave behind a track of past locations.

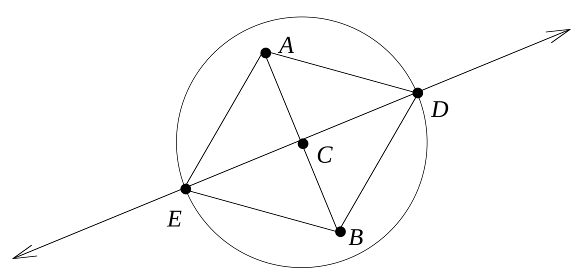


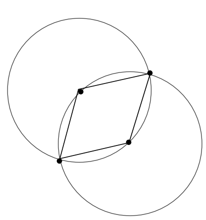
An experiment: Create $\overline{AB}$ with midpoint C. For B and C, turn on the Trace feature. Now "draw" your initials with B. C leaves a tracing half the size but similar to the tracing left by B.

Problem 23 *(Student page 94)*

A rhombus can be constructed as a parallelogram with all sides congruent.

It is even easier to construct if you make use of the fact that the diagonals of a rhombus are perpendicular and bisect each other. Draw a segment, $\overline{AB}$, construct its perpendicular bisector, and place a point D anywhere on that perpendicular bisector. Now center a circle of radius CD at the midpoint C. $ADBE$ is a rhombus.



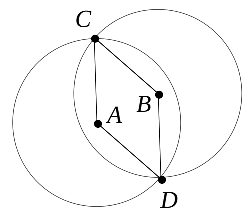


An even easier way you can start is to use the two circles you constructed in Problem 13 and connect their intersections to their centers. This rhombus will always have the shape of two equilateral triangles glued together along a side: a fine solution, but limited to one shape.

Having discovered the two-circle idea, though, you can then loosen it up to allow more flexibility in the shape of the rhombus. Construct two circles that have the same

What happens when you pull the two circles far enough apart so that they no longer intersect? Can you find a way of constructing this figure so that this cannot happen?

radius. Position them so that they intersect, and connect their intersection points to their centers. All four segments are congruent (why?), so the resulting figure is a rhombus.

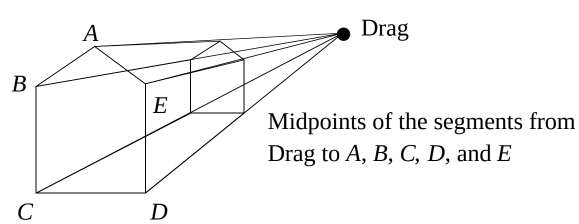


A and B are centers of congruent circles. C and D are the points where the circles intersect.

Problem 24 *(Student page 95)* The construction described for Problem 15 ensures that all three sides are congruent by building them as radii of congruent circles.

Problem 25 *(Student page 95)* Perhaps the easiest way to create the perspective house is to draw the "front" of the house ($\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, and $\overline{EA}$), pick an arbitrary point F (we've changed its label to *Drag*), and connect that new point to each vertex of the house. Points at the "back" of the house can be constructed as midpoints of these segments. If your software can change the color or thickness of a line, that may help highlight parts of the drawing.

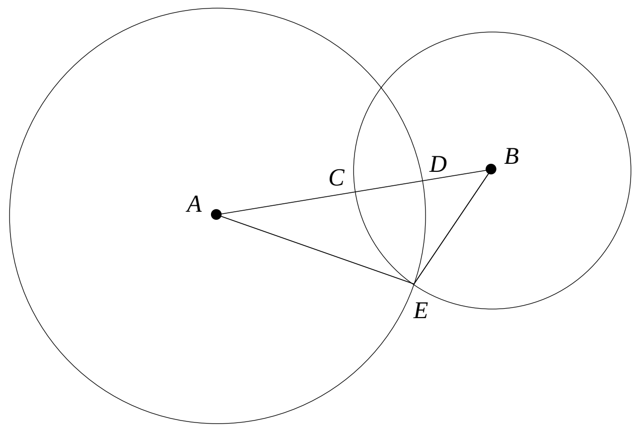
This procedure guarantees that the front and back of the house are "geometrically similar"—the same shape (regardless of size). Their corresponding angles will have the same measure, and their sides will be in proportion.



Problem 28 *(Student page 97)* This problem revisits ideas you've seen in ruler and compass constructions in Problem 9 of Investigation 1.5.

Create $\overline{AB}$ at any length, find its midpoint C, and use C as a point on the circle to create a circle centered at B. Find the midpoint D of $\overline{BC}$ and create a new circle,

centered at A, with radius AD. These two circles intersect in two points. Call one of the intersection points E. This construction assures that $\frac{BE}{AB} = \frac{1}{2}$, and that $\frac{AE}{AB} = \frac{3}{4}$, as desired.



WARM-UPS

Some "invariants" are built-in (for example, the fact that the units digits in the pairs of numbers in Part *b* add up to 10 before squaring); others are discovered as consequences of the built-in properties. When you are asked to *look for* invariants, that generally means only the consequences. But when you try to *explain why* things are invariant, you'll need to pay attention to the built-ins, too.

Problem 1 *(Student page 98)*

a. All squares of these numbers will end in 25. In fact, even the hundreds digit of the squares is partially determined: it must be one of 0, 2, or 6.

b. Here are two theorems you might have discovered. (They both really say the same thing! And there are other ways of wording this same discovery, too.)

THEOREM *Square Numbers (1)*

If the units digits of a pair of numbers add up to 10, then the units digits of their squares will be equal.

THEOREM *Square Numbers (2)*

If the sum of two numbers is a multiple of 10, then the difference of their squares is also a multiple of 10.

The algebraic proof of the second theorem begins by restating the premise

$$a + b = 10n,$$

and then expressing one of the two numbers in terms of the other: $b = 10n - a$. The claim is that $b^2 - a^2 = (10n - a)^2 - a^2$ is a multiple of 10. Finish the proof.

c. Because no *rule* is given for the set {1, 4, 7, 10, ...}, it *could* be *any* set containing the given numbers. But, if we assume these elements are representative, then the set appears to contain "every third number, starting with 1." It would be tedious to check 301 that way, but "every third number starting with 0" is simply the multiples of 3. So another way to describe this set is "numbers of the form $3n + 1$." Then it is easy to see that 301 is a number in the set.

Extra Challenge The product of any two of the numbers belongs to the set, as does the sum of any four of them. The difference of any two numbers does not. These statements are easy to prove algebraically. Here's one of the proofs. You can complete the others.

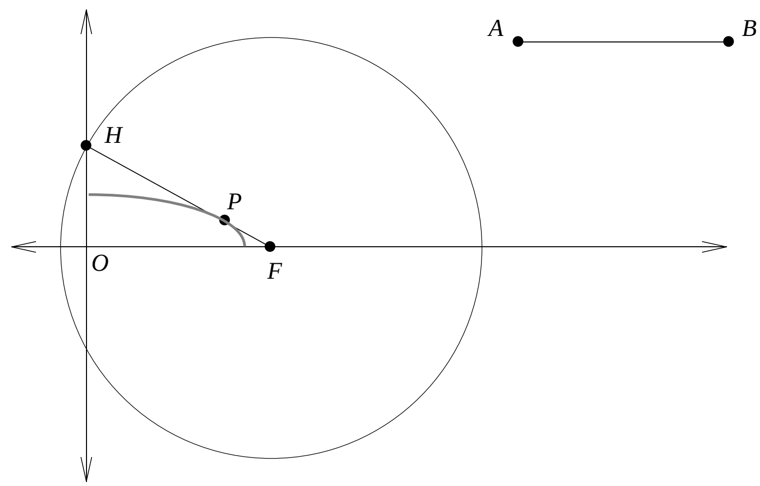
$$(3a + 1)(3b + 1) = 9ab + 3a + 3b + 1 = 3(3ab + a + b) + 1$$

Problem 2 *(Student page 99)* Here is a model in geometry software. The ground

Warm-Ups (continued)

and walls are built as two perpendicular lines. $\overline{AB}$, off in a corner, is the fixed length of the ladder. The foot F of the ladder is a point on the "ground," and a circle with center F and radius AB intersects the "wall" at H, the head of the ladder $\overline{FH}$. As F moves along the ground, the ladder (a radius of the circle) remains a fixed length, so H moves appropriately along the wall.

If you Trace the path of person P standing on the ladder, that path looks like (and can be proved to be) a quarter of an ellipse. If P is halfway up the ladder, the path looks (and can be proved to be) circular. It is a nice challenge, and not too difficult, to figure out how this last fact is equivalent to saying that the midpoint of the hypotenuse of a right triangle is equidistant from all three vertices!



Some things that change:

- Position of the foot of the ladder
- Position of the head of the ladder
- Slope of the ladder
- Area of the triangle under the ladder (When is it greatest? Smallest?)
- The sum of distances: $HO + OF$ (When is it greatest? Smallest?)

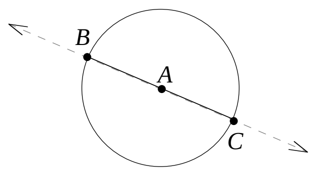
Some things that don't change:

- The right angle between the wall and the ground
- The sum of angles: $\angle OHF + \angle OFH$

WARM-UPS (continued)

- The length of the ladder, which is the hypotenuse of the triangle
- The distance of the midpoint of the ladder to O
- The sum of squared distances: $(HO)^2 + (OF)^2$.

To construct a diameter for an existing circle: **Construct a line using the center of the circle and a point on the circle. The two points of intersection of the line and the circle are the endpoints of a diameter.**



Given a circle with center A, $\overleftrightarrow{AB}$ determines C. $\overline{BC}$ is a diameter.

Problem 3 *(Student page 99)* Make sure to *construct* the diameter, rather than just draw a segment that looks right. The ratios $\frac{circumference}{diameter}$ and $\frac{diameter}{circumference}$ are constant, while the others vary.

DEFINITION

The mathematical constant π is *defined* as the ratio $\frac{circumference}{diameter}$.

You probably also noticed the qualitative invariant—“the area (invariably) grows as the diameter grows”—but may have considered it too trivial to mention. Things like that should not be dismissed *too* lightly. After all, the area of the triangle under the ladder in Problem 2 did *not* invariably grow (or always shrink) as the distance OF increased.

Problem 4 *(Student page 99)*

a. $m\angle EDF$ is not invariant. (When does it seem largest?)

b. $DE + DF$ is not invariant, but the Triangle Inequality, $DE + DF > EF$, still holds.

c. The perimeter of $\triangle DEF$ is not invariant.

d. The area of $\triangle DEF$ appears invariant. The area of a triangle is given by $\frac{1}{2}\ base \times height$. In this case, EF is constant as a base, and the height is always the (unchanging) distance between the two parallel lines.

e. The sum of the angles in a triangle is invariant.

Problem 5 *(Student page 100)*

a. Alternate interior angles are equal in measure. For example, $m\angle CDE = m\angle FED$.

b. The sum of the measures of adjacent angles along $\overleftrightarrow{AB}$ (for example, $\angle AED$ and $\angle FED$) is 180°, as is the sum of the angle measures in a triangle. As a

consequence (figure out how!), an exterior angle of a triangle has a measure equal to the sum of the two opposite interior angles of the triangle.

The "same-side interior angles" are a bit less camouflaged if either $\overline{DE}$ or $\overline{DF}$ is erased.

For each of these, other answers, which may be much more complicated, are possible.

Also, if two angles lie between the parallels and on one side of the transversal (for example, $\angle AED$ and $\angle CDE$ or $\angle AFD$ and $\angle CDF$), they are supplementary.

c. $\angle DFB$ is always larger than $\angle DEB$; $\angle DEA$ is always larger than $\angle DFA$.

Problem 6 *(Student page 100)*

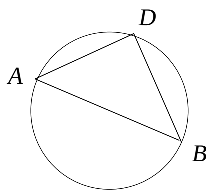
a. $CD = DB = \frac{CB}{2}$

b. $\text{Area}(\triangle ADC) = \text{Area}(\triangle ADB)$

$\text{Area}(\triangle ADC) = \frac{1}{2}\text{Area}(\triangle ABC)$

c. Here's a curious one: $CA + AB - 2AD \geq 0$. Can you prove it?

NUMERICAL INVARIANTS



Problem 1 *(Student page 101)* In the sketch in the margin, with $\overline{AB}$ a diameter of the circle, we see that

- $m\angle ADB$ is constant (90°);
- $AD + DB > AB$;
- $(AD)^2 + (DB)^2 = (AB)^2$;
- $m\angle BAD + m\angle ABD$ is constant (90°);
- $AB \geq AD$, $AB \geq BD$.

Problem 2 *(Student page 101)* Using the same sketch, we see that

- $m\angle ADB$ is constant (90°);
- ratio of *any* two segment lengths (for example, $\frac{AB}{AD}$) is constant;
- $\frac{\text{Circumference}(\odot)}{AB}$ is constant (π);
- $\frac{\text{Area}(\triangle ABD)}{\text{Area}(\odot)}$ is constant;
- $m\angle DAB$ is constant;
- $m\angle DBA$ is constant.

Problem 3 *(Student page 102)*

For each of those, other answers, which may be much more complicated, are possible.

- $d - c = 6$
- $e + f = 10$
- $x \times y = 5$
- $a + b = 180$
- $w \times z = 16$
- $\frac{h}{g} = 4$ or $\frac{g}{h} = \frac{1}{4}$
- $\frac{n}{m} = 2.5$ or $\frac{m}{n} = 0.4$
- $q - p = 90$

Problem 4 *(Student page 102)*

a.
- c, d same direction; subtraction gives a constant.

- e, f opposite directions; addition gives a constant.
- x, y opposite directions; multiplication gives a constant.
- a, b opposite directions; addition gives a constant.
- w, z opposite directions; multiplication gives a constant.
- g, h same direction; division gives a constant.
- m, n same direction; division gives a constant.
- p, q same direction; subtraction gives a constant.

b. When the difference between two variable quantities is constant, then as one increases, the other must also increase, and by the same amount. When two quantities have a constant ratio, then as one increases, the other must also increase, and at the same *rate*. If the sum or product of two quantities is constant, then one must decrease as the other increases.

These observations lead to a useful strategy. When two quantities vary in opposing directions, one might check their sum or product to see if it stays still enough to suggest a usable pattern. And when, as in Problem 8 below, two quantities vary in the same direction, one might look for patterns in their difference or ratio before going on to look for more complex relationships.

Problem 5 *(Student page 102)* The ratio of the two values ($\frac{a}{b}$ or $\frac{b}{a}$) is a constant and the values in column a of this table are increasing as the values in column b are decreasing. This contradicts the rules stated in the Problem 4b. The statement should then be changed to:

- The *absolute values* change in the *same direction*: both increase or both decrease.
- The *absolute values* change in *opposite directions*: one increases as the other decreases.

Problems 6–7 *(Student pages 102–103)* Be able to explain your rule to someone else.

Problem 8 *(Student page 103)* Without some special strategies, it can be quite difficult to find a pattern in this table. Here are two ways to think about this problem.

The values vary in the same direction—as r increases, so does s. Without even performing the calculations, you can see that their difference is not constant (271 is much farther away from 108 than 114 is from 45), so let's look at their ratios. Using a calculator or Logo, we compute these ratios: 2.533333333 . . . , 2.525, 2.520833333

. . . , 2.513888889 . . . . Using a intermediate value might be tempting, but the mean of these four numbers, 2.523263889 . . . , seems far too "messy" a number for a table that contains only integers or integers plus $\frac{1}{2}$.

In this case, the *ratios* are also extremely close to "an integer plus one half." So we decide to see what happens if we ignore the messy part and pretend we'd seen ratios of *exactly* 2.5 each time. So we perform a new experiment, multiplying r by 2.5 to see whether these new numbers give us some insight into the pattern.

r	$2.5r$	s
45	112.5	114
60	150	151.5
72	180	181.5
108	270	271.5

From this table, we find the relationship we are looking for:

$$s = 2.5r + 1.5.$$

Although this kind of strategy doesn't *guarantee* success, numerical patterns often do yield pretty quickly to this kind of experimentation.

An alternative approach, of course, is to graph the data and see what relationship the graph suggests.

Problem 9 *(Student page 103)* If C remains between (fixed) points A and B, then the closer it is to A, the farther it will be from B: the lengths AC and CB vary in opposite directions. Their sum $AC + CB$ is constant and equal to AB.

Problem 10 *(Student page 103)* If C is on $\overleftrightarrow{AB}$ but *not* between A and B, then as it gets farther from either point, it also gets farther from the other: the lengths AC and CB vary in the same direction. Now, the *difference* between AC and BC is constant and equal to AB.

Problems 11–12 *(Student page 104)* The cutting experiment shown in the Student Module gives the following result: Every n-gon can be cut up into $(n - 2)$ triangles. (For the moment, we take this without proof, although you can probably find some very convincing logical reasons.) Each triangle contains 180°-worth of angle, so the angle sum of any n-gon is $(n - 2) \times 180$.

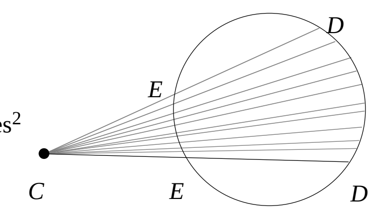
The proof of this theorem is found in the *Connected Geometry* module *A Matter of Scale*.

Problems 13–14 *(Student page 105)* The two lengths vary in opposite directions. Their *sum* cannot be constant (because they add up to the full length of the chord, which changes). Checking the product to see where *that* might lead produces the remarkable result that $CE \times CD$ is constant for a fixed point C.

The constant product associated with a given point C is traditionally called the *power* of the point, and it depends on the distance of that point from the circle. When C is very close to the circle, at least one of the two distances along the chord from C to the circle will be very small, and so the product will be small. (When C is *on* the circle, one distance is always 0, and so the product is 0.)

You might wonder what happens if point C is outside the circle, as shown below. The two distances along the same line to the circle would then be CE and CD. How do those quantities vary as D moves, and is there another invariant?

$m\ \overline{CD} = 1.58$ inches
$m\ \overline{CE} = 0.74$ inch
$(m\ \overline{CE})(m\ \overline{CD}) = 1.17$ inches^2



Problems 15–16 *(Student page 105)* Here are some invariants that you "know about before you start" (because you deliberately built them in).

- $CD = DA$ (D is a midpoint)
- $AE = EB$ (same reason)
- $\frac{AD}{AC} = \frac{1}{2}$
- $\frac{AE}{AB} = \frac{1}{2}$

Here is an invariant that *you* did not deliberately build in, but that the people designing the software did deliberately build in. You may have ignored it as too boring to list, but it is important.

- As you drag one vertex of the triangle, the other two vertices do not move, and therefore the length of the opposite side does not change.

Here are some invariants that are not "deliberately" built in by anybody, but are natural *consequences* of the ones listed above.

- CB and DE vary in the same direction: $\frac{CB}{DE} = 2$.

NUMERICAL INVARIANTS (continued)

- $\frac{\text{Area}(\triangle ABC)}{\text{Area}(\triangle AED)} = 4$
- $\overline{DE}$ is parallel to $\overline{CB}$ (same slopes).

MIDLINE CONJECTURE

A segment connecting the midpoints of two sides of a triangle is parallel to the third side and half its length.

Problem 17 *(Student page 106)* There are many invariants to find. Here are a few:

- $\frac{AE}{AD}$ is invariant, as are $\frac{DE}{AD}$ and $\frac{AE}{DE}$. The fact that these are invariant also makes their sums, differences, products, squares, and so on, invariant.
- $AD + DC$ is invariant. (It equals AC.)
- $\frac{BA}{EA} = \frac{CA}{DA} = \frac{CB}{DE}$ (the "Parallel Theorem")
- $\frac{\text{Area}(\triangle ABC)}{\text{Area}(\triangle AED)} = \left(\frac{BC}{ED}\right)^2$
- Never overlook the obvious: The lengths of the sides of the large triangle, $\triangle ABC$, don't change as you move D.
- The problem didn't ask about angles, but they, too, do not change. And the *shape* of $\triangle ADE$ doesn't change, even though its size does.

Sometimes referred to as the "Parallel Theorem," the equality of these ratios is proved in the *Connected Geometry* module *A Matter of Scale*.

Problem 18 *(Student page 107)*

a. No

b. Yes; they are equal, so the ratio is 1:1.

c. No

d. Yes; they are equal, so the ratio is 1:1.

Angle measure is invariant.

Problem 19 *(Student page 107)* The midpoint of the hypotenuse is equidistant (the same distance) from all three vertices.

Problem 20 *(Student page 107)*

a. The path of P is the perpendicular bisector of $\overline{AB}$.

b. The path of P is a circle of radius 5 centered on A.

c. P lies on an ellipse whose foci are A and B.

d. P is on a circle with diameter $\overline{AB}$.

e. P is on a circle that has points A and B as endpoints of an arc (or defining a central angle) of 60°. (See Problem 14 in Investigation 1.1 for a related experiment with inscribed angles.)

f. P is on one of two rays with endpoint B that meet $\overline{AB}$ at 30°.

Solutions

Investigation **1.16**

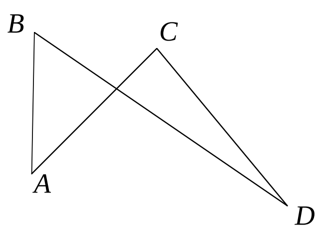
Student Pages 108–116

SPATIAL INVARIANTS

Problems 1–2 *(Student page 108)* The triangle invariably contains a hexagon, but otherwise the strict doubling pattern is not reliable. However, there is a looser pattern: for an n-gon (other than the triangle), the maximum number of sides for the inside figure is $2n$, and the minimum is $2n - 2$.

The inner polygons can never be regular. This is easiest to see when the outer polygon has an odd number of sides: the opposite sides of the inner polygon cannot be parallel (why?), which would be necessary if the polygon were to be regular.

A figure like the one below is not technically a quadrilateral because it is self-intersecting. This is what we mean by a quadrilateral that is not "normal."



Problem 3 *(Student page 109)*

a. If the outer figure is any "normal" quadrilateral, the inner figure will have four distinct vertices, so it must be a quadrilateral. (This will be true "most" of the time even when the outer figure is *not* "normal." Can you find a not-so-normal situation when the inner figure is *not* a quadrilateral?)

b. At this stage, all you probably have is observation, but the inner figure appears to be a parallelogram.

Problem 4 *(Student page 109)* The concurrence is not invariant.

Problem 5 *(Student page 110)* All regular polygons with an even number of sides will have diagonals that concur at the polygon's center. Other concurrences will also occur as the number of sides increases.

Geometric software will allow you to draw "quadrilaterals" like this one, so you may want to investigate figures like this.

In fact, when concurrence occurs, it should be considered enough of a surprise to check it for invariance. If it *is* invariant, that is always worthy of special mention!

Problem 6 *(Student page 110)* It may take patience to adjust your figure, but there are many pentagons whose perpendicular bisectors all concur at a single point. If you're working with software, one way to help adjust the points is to draw a circle at some convenient place on your screen and then move each vertex of the pentagon onto the circle.

Problem 7 *(Student page 111)* It may take patience to adjust your figure, but there are many pentagons whose angle bisectors all concur. Again, a circle can help, but this time the circle must be *inside* the pentagon, just touching (tangent to) each side.

Problem 8 *(Student page 111)* There is no way to adjust a triangle so that the perpendicular bisectors are *not* concurrent!

The point of concurrency of perpendicular bisectors is called the *circumcenter*: the center of a circle *circum*scribed about the triangle. The point of concurrency of angle bisectors is called the *incenter* of a triangle: the center of a circle *in*scribed in the triangle.

Problem 9 *(Student page 111)* Again, concurrence of angle bisectors is automatic (unavoidable, invariant) for all triangles. The point of concurrency is called the *incenter* of a triangle. All the points on an angle bisector are the same distance from the sides of the angle.

Problem 10 *(Student page 111)* For perpendicular bisectors and angle bisectors all to concur at the same point, the triangle must be regular (equilateral).

Problem 11 *(Student page 112)*

It is not at all clear what one might mean by "concurrency of perpendicular bisectors" for a square.

THEOREM ***Concurrency of Perpendicular Bisectors, Regular Polygons***

For regular polygons (other than a square), the perpendicular bisectors of the sides are concurrent.

In some (but not all) nonregular polygons, all perpendicular bisectors can be concurrent.

Symmetry gives a good informal explanation of why this *should* be so. A regular polygon is one that, if rotated just the right amount around its "center," looks just as it did before. That center is the same distance from every vertex of the polygon. Finish the explanation!

Problem 12 *(Student page 112)* Experimentation shows (and symmetry, again, helps prove) the following theorem:

It is not at all clear what one might mean by "concurrency of angle bisectors" for a square.

THEOREM ***Concurrency of Angle Bisectors, Regular Polygons***

For regular polygons (other than a square), the angle bisectors are concurrent.

In some (but not all) nonregular polygons, all angle bisectors can be concurrent.

Problem 13 *(Student page 112)* The perpendicular bisectors of these sides are concurrent at the center of the circle. Here's why:

Perpendicular bisectors of a segment are lines containing all the points that are equidistant from the endpoints of the segment. If the endpoints of the segment are on a circle, then the center of the circle (which is certainly equidistant from those endpoints) must lie on the perpendicular bisector of the segment.

SPATIAL INVARIANTS (continued)

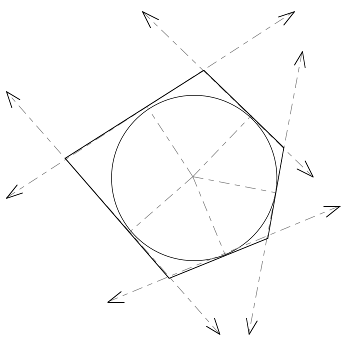
A chord of a curve is a segment whose endpoints are on the curve.

THEOREM ***Perpendicular Bisector of a Chord***

The perpendicular bisector of every chord of a circle must pass through the center of the circle.

The sides of the inscribed pentagon in this problem are chords of a circle. Therefore, the perpendicular bisectors of the sides must pass through the center.

The *angle* bisectors of the inscribed pentagon show no particular relationship.



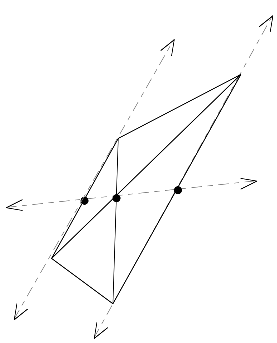
The lines are tangent to the circle because each is perpendicular to a radius at a point on the circle.

Problem 14 *(Student page 112)* In this situation, all the angle bisectors are concurrent at the center of the circle, but the perpendicular bisectors show no special relationship.

Here's an easy way to construct a figure with which you can experiment: draw a circle, construct a few radii, and then construct perpendiculars at the points on the circle. Those tangent lines can define the sides of the polygon. This construction can help you prove the following theorem:

THEOREM ***Angle Bisector, Tangents***

The bisector of an angle formed by two tangents of a circle must pass through the center of the circle.

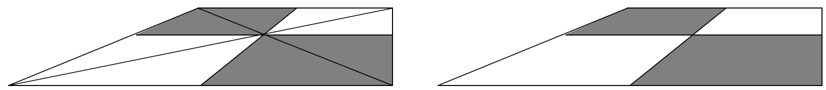


Thinking of the triangle suggested by the trapezoid may make the collinearity more expected.

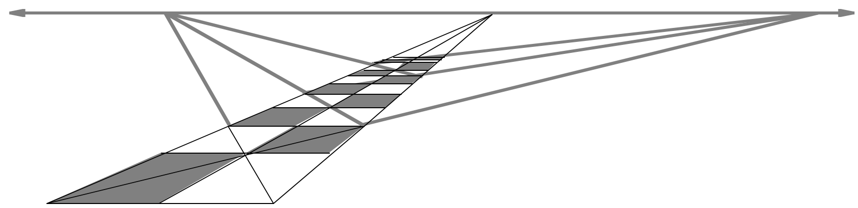
Problem 16 *(Student page 113)* The center of a circle is a point equidistant from all the points on the circle. Thus, when any circle is placed so that it passes through two points, then the center is equidistant from those two points. Perpendicular bisectors of a segment are lines containing *all* the points that are equidistant from the endpoints of the segment. Therefore, the center of any circle passing through the points must lie on that perpendicular bisector. This is really just the Perpendicular Bisector of a Chord Theorem in a different form.

Problem 17 *(Student page 113)* The midpoints of the parallel segments are built to be collinear (on the same line) with the endpoints of the segments, but they turn out to be collinear with the intersection of the diagonals as well.

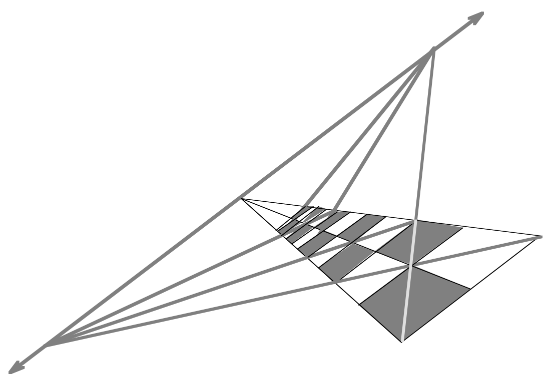
This and other collinearities are much used by artists when they want to make even spacing in perspective drawings.



How to draw a (square) checkered floor in perspective

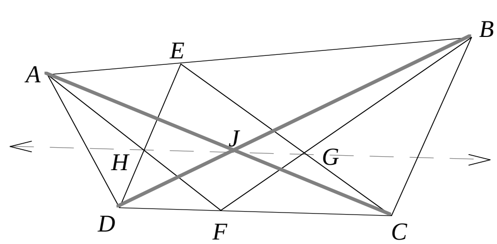


A checkerboard road



A "road" on a tilt

Problem 18 *(Student page 113)* There are many *non*surprising collinearities: for example (A, E, B), (E, G, C), and (B, J, D). But the collinearity of H, J, and G is most surprising.



Problem 19 *(Student page 114)* Point P is constructed to be equidistant from A and B and equidistant from A and C. That makes it also equidistant from B and C, so P must lie on the perpendicular bisector of $\overline{AB}$.

THEOREM ***Concurrency of Perpendicular Bisectors, Triangles***

The perpendicular bisectors of the sides of any triangle are concurrent. (This is really a consequence of the Three-Point Circle Theorem.)

Problem 20 *(Student page 115)* If all the points on the angle bisector of $\angle ABC$ are the same distance from sides $\overline{AB}$ and $\overline{BC}$ and all the points on the angle bisector of $\angle CAB$ are the same distance from sides $\overline{AC}$ and $\overline{AB}$, then the intersection point of those bisectors must also be the same distance from sides $\overline{AC}$ and $\overline{BC}$. That puts it on the angle bisector of $\angle ACB$ as well.

THEOREM ***Concurrency of Angle Bisectors, Triangles***

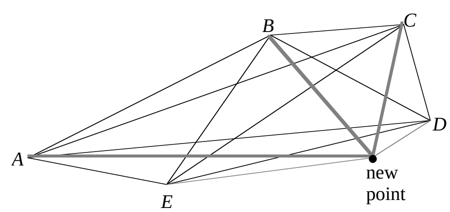
The angle bisectors of any triangle are concurrent.

Feel *very* proud of yourself if you've made a dent in this problem, and don't feel bad if you have *not* made much progress. This problem is generally considered *extremely* difficult.

The "explanation" makes several assertions without proving they are true. The claim about the diagonals of a pentagon is one of these. Can you find the others? Can you prove or disprove them?

Problem 24 *(Student page 116)* Here's a claim and part of a strategy for proving or disproving it. Our "solution" leaves a *lot* of room to do better.

Original statement: "In *any* hexagon with all diagonals drawn in, there can be *at most* one concurrence of three diagonals." **Claim:** The statement is *true*. **Explanation:** In any *pentagon*, it is impossible to get a concurrence of three diagonals: at most two diagonals will intersect in any point. (Is this really true, or does it merely look true? Can you *prove* it true or false?) Draw a pentagon and its diagonals. Now, we'll try to convert the pentagon into a hexagon by adding a new point somewhere and drawing in the new sides and diagonals, as in the picture below. If it is possible to find a place for that new point that allows more than one of the new diagonals to pass through the already existing intersections of other diagonals, then the statement is false. But that is not possible: only one of the new diagonals *can* pass through already existing intersections. So the original statement is true.



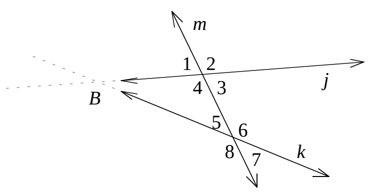
Parallel Lines

Problem 1 *(Student page 117)*

a. The interior angles are "inside" the parallel lines. "Alternate" means they are on opposite side of the transversal, and on opposite "banks of the river" that is bounded by the parallels.

b. Alternate exterior angles are two angles that are "outside" the parallel lines, and on opposite sides of the transversal, such as $\angle 2$ and $\angle 8$.

Problem 2 *(Student page 117)*



a. The pairs ($\angle 4$, $\angle 8$), ($\angle 1$, $\angle 5$), ($\angle 2$, $\angle 6$), and ($\angle 3$, $\angle 7$) are all corresponding angles.

b. Corresponding angles are on the same side of the transversal, and are on the "same" sides of the two lines in question.

Problem 3 *(Student page 118)* We know of no widely-used name for angles that are related in this way. $\angle 2$ and $\angle 7$ are certainly "exterior" angles. Perhaps "same-side exterior" would be a good name for them. This is compatible with our use of "side-side interior" for a pair of angles such as $\angle 3$ and $\angle 6$.

Problem 4 *(Student page 118)* If we number the angles as they are in Problem 1, then we can say that pairs of angles like $\angle 2$ and $\angle 3$, or $\angle 7$ and $\angle 8$ are supplementary.

Why are they called "vertical angles" if they aren't always "vertical"? These angles are called *vertical* because they are joined in a natural way at their *vertices*. The two meanings of vertical come from the two meanings of *vertex*: the highest point (which gives rise to the common up-and-down meaning of vertical), and the "point" of an angle. Jagged mountain peaks show the natural connection of the two meanings.

Vertical angles: There are other pairs of angles that always have equal measures. If you focus on just one intersection point and the angles around it, you will see pairs like ($\angle 2$, $\angle 4$) and ($\angle 1$, $\angle 3$) and ($\angle 5$, $\angle 7$) that always have equal measures. These important pairs are called *vertical angles*. You can show that they *must* have equal measure: $m\angle 1 + m\angle 2 = 180 = m\angle 3 + m\angle 2$, so $\angle 1 = \angle 3$.

a. Moving the transversal does not affect the sum $m\angle 3 + m\angle 6$; the sum will remain constant. The same is true for various other sums and differences, including $m\angle 4 + m\angle 5$, $m\angle 1 + m\angle 8$, $m\angle 2 + m\angle 7$, $m\angle 3 - m\angle 7$, and so on.

b. Although moving one of the lines will *change* the sums mentioned above, $m\angle 3 + m\angle 6 = m\angle 1 + m\angle 8$, no matter how things are moved, and also $m\angle 4 + m\angle 5 = m\angle 2 + m\angle 7$. Why?

Problem 5 *(Student page 118)* Numbering the angles the same way, the angle sums mentioned above are also invariant here. Also, corresponding angles have equal

measure, as do alternate interior and alternate exterior angles, so

$$m\angle 2 = m\angle 4 = m\angle 6 = m\angle 8,$$

$$\text{and}\quad m\angle 1 = m\angle 3 = m\angle 5 = m\angle 7.$$

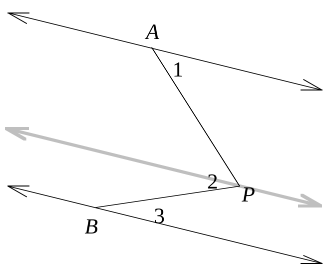
Looking at the angles on only one side of the transversal, the interior angles (such as $\angle 3$ and $\angle 6$) and the exterior angles (such as $\angle 2$ and $\angle 7$) have angle measures whose sum is 180°.

Problem 6 *(Student pages 118–119)*

a. Yes, they must be parallel for the angles to be equal in measure.

b. If both pairs of lines are parallel, $m\angle 2 = m\angle 3$ at all times.

c. If lines a and b are not parallel, the above relationships do not hold.



$m\angle 1 + m\angle 3 = m\angle 2$

Problem 7 *(Student page 119)* Perhaps the most interesting invariant in this situation is that the measure of the angle at P ($\angle 2$) equals the sum of the measures of the angles that the two segments from P make with the parallel lines ($\angle 1$ and $\angle 3$). One way to explain this is to draw a third parallel through P and look at the way that line cuts $\angle 2$.

Problem 8 *(Student page 119)*

THEOREM *Isosceles Triangles*

The base angles of isosceles triangles are equal in measure (congruent).

a. This experiment allows you to conjecture that when $BC = CD$ (that is, when $\triangle BCD$ is isosceles), $m\angle CBD = m\angle CDB$, and when $CD = DE$ (that is, when $\triangle CDE$ is isosceles), $m\angle DCE = m\angle DEC$. Paper folding suggests one way to begin a proof for this theorem. Fold an isosceles triangle so that its two congruent sides match up. See what happens to the base angles, and argue why this *must* happen.

Do you consider all the steps in this reasoning *reliable*, or do you think some of them are unproved claims?

b. If $\overline{BC}$ and $\overline{DE}$ were parallel, then alternate interior angles $\angle BCD$ and $\angle CDE$ would be congruent. If that were true, isosceles triangles BCD and CDE would have the same vertex angles, which would give them the same base angles. (Why?) And if base angles $\angle BDC$ and $\angle ECD$ were congruent, then $\overleftrightarrow{AE}$ and $\overleftrightarrow{AD}$ would have to be parallel. (Why?)

Problem 10 *(Student page 120)* A thorough answer will explain how to change part of the question or statement to make it possible.

a. Impossible: Angles $\angle 3$ and $\angle 4$ are supplementary; the sum of their measures is 180°. Angles $\angle 3$ and $\angle 6$ can sum to 180° only if $m\angle 6 = m\angle 4$. As alternate interior angles, $\angle 6$ and $\angle 4$ can be congruent only if n and p are parallel.

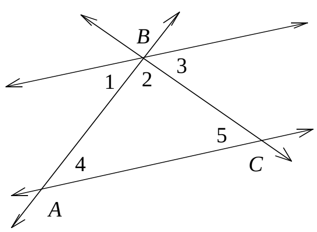
b. Possible: Alternate interior angles are congruent if the lines are parallel.

c. This is possible only if line q is perpendicular to both lines n and p.

d. Impossible: $\angle 4$ and $\angle 2$ are vertical angles, and are therefore congruent. The same is true for $\angle 5$ and $\angle 7$. The sums of equals are equal.

e. Impossible: If $\overleftrightarrow{AB}$ is parallel to $\overleftrightarrow{CD}$, then $m\angle JGB + m\angle GJD = 180^\circ$. That means $m\angle JGH + m\angle GJH = 90^\circ$. The sum of the measures of the angles in a triangle is 180°; therefore $m\angle GHJ$ must be 90°.

Problem 11 *(Student page 120)* Here are some steps in the argument. Is anything important left out?



- $m\angle 1 + m\angle 2 + m\angle 3 = 180^\circ$ because these three angles all lie along a straight line.
- $m\angle 1 = m\angle 4$ because $\angle 1$ and $\angle 4$ are alternate interior angles. $m\angle 3 = m\angle 5$ for the same reason.
- Substitute $m\angle 4$ and $m\angle 5$ into the first equation to get $m\angle 4 + m\angle 2 + m\angle 5 = 180^\circ$.

Midlines and Marion Walter's Theorem

Problem 1 *(Student page 122)* Possible conjectures:

The point of concurrency of the three medians is called the *centroid*.

- The three medians are concurrent.
- Each median divides the whole triangle into two smaller triangles that are equal in area.
- The three medians divide the original triangle into six small triangles; all six have equal area.
- There are various congruent vertical angles.
- The centroid divides each median into two parts whose lengths are in a 2:1 ratio.
- The centroid always falls *inside* the triangle.

This means that each median gets split into $\frac{1}{3}$ and $\frac{2}{3}$ pieces.

Problem 2 *(Student page 122)*

a. A midline's length is half that of the parallel side.

b. Each small triangle has $\frac{1}{4}$ the area of the original triangle.

Other possible observations:

- Each midline appears to be parallel to a side of the triangle.
- Corresponding vertex angles in the small triangles are equal in measure. (Use alternate interior angles, and assume that midlines are parallels.)
- Each vertex angle in the small triangles is equal in measure to a vertex angle of the original triangle.
- The perimeter of a small triangle is half the perimeter of the original.

Problem 3 *(Student page 122)* Possible observations or invariants:

That is, the area is split into $\frac{1}{3}$ and $\frac{2}{3}$ pieces.

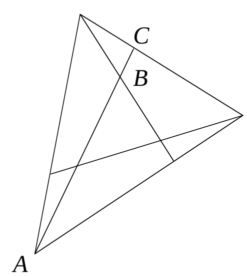
- One internal segment subdivides the original triangle into two parts whose areas are in a 2:1 ratio.
- The internal segments are subdivided into three pieces which always have length ratios of 3:3:1 (or 1:1:$\frac{1}{3}$).
- The central shape is always a triangle.
- The area of the central triangle is $\frac{1}{7}$ the area of the original triangle.
- The small corner triangles are each $\frac{1}{3}$ the area of the central triangle.

Problem 4 *(Student page 123)* Possible observations or invariants:

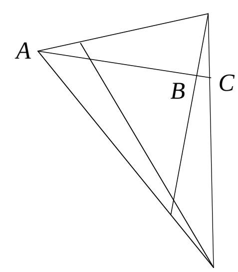
- The internal segments are subdivided into five pieces which have interesting invariant length ratios. For instance, the first and last piece are always in a ratio of 3:1.
- The central shape is always a hexagon.
- The ratio between the area of the central hexagon and the area of the original triangle is invariably 1:10.

Problems 5–7 *(Student page 123)* There are many possible patterns to notice. Here is one about the lengths of the subdivisions of the internal segments:

When the triangle's sides were bisected, the internal lines were medians. The short-end part of the median was $\frac{1}{3}$ the length of the whole median. Here are the next two cases:



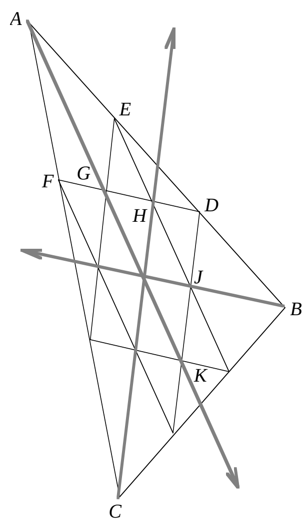
The triangle's sides are tri-sected. Now $\overline{BC}$ is $\frac{1}{7}$ the length of $\overline{AC}$.



The triangle's sides are "4-sected." Now $\overline{BC}$ is $\frac{1}{13}$ the length of $\overline{AC}$.

Below is a table that summarizes the observations. Do these form a *reliable* pattern? That is, can you find a pattern that would allow you to predict the value of $\frac{BC}{AC}$ for 5-section or 100-section and can you find a way to prove that "real life" will *follow* that pattern?

"*n*-section"	$\frac{BC}{AC}$
2	3
3	7
4	13
⋮	⋮
n	???



Problem 8 *(Student page 123)* There is a lot to find in this figure:

- The inner shape will always be a six-pointed star made of six triangles surrounding a hexagon.
- Opposite sides of the hexagon will always be parallel.
- Opposite sides of the hexagon will always be the same length.
- Opposite vertices of the hexagon are collinear with a vertex of the triangle, and the line through them bisects a side of the triangle.
- Diagonals connecting the opposite vertices of the hexagon are concurrent.
- The six triangles (like $\triangle GEH$ and $\triangle HDJ$) that surround the hexagon to form the points of the star (and the six triangles that one sees inside the hexagon when the medians of $\triangle ABC$ are sketched in) are all congruent—the same size and shape.
- The area of one of these little triangles is $\frac{1}{27}$ the area of the original triangle.
- Triangles like $\triangle AGE$ and $\triangle AGF$ are *not* the same shape but have equal area, twice the area of the little triangles.
- The area of the hexagon will always be $\frac{6}{27}$ the area of the original triangle.

A Folding Investigation

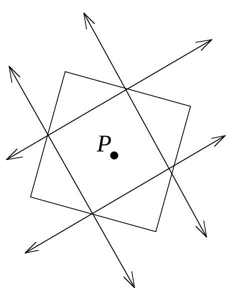
Problem 9 *(Student page 127)* The segment that connects the marked point to the chosen corner is being folded in half, so the crease is along the perpendicular bisector of that segment.

Problem 10 *(Student page 129)* It *is* possible to find an example that is not a hexagon.

Problem 11 *(Student page 129)* Here are some ways to think about these problems:

- **a.** If the lines of the creases were extended beyond the edges of the paper, they would surround a quadrilateral (four lines, four sides). All that the edges of the paper can do is "clip" corners off of that quadrilateral, so the region of paper enclosing P cannot have more than eight sides.
- **b.** Each fold produces a side for the region enclosing P, so there must be at least four sides to the region.

Putting *P* right in the center makes a square inside. Moving *P* will "clip" a corner or two.



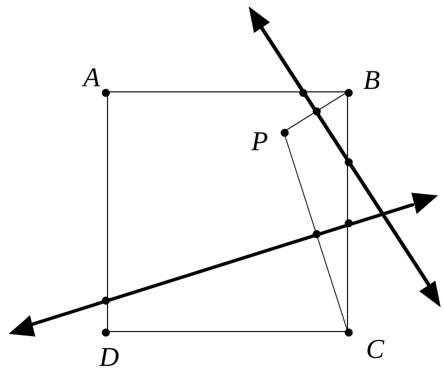
Problem 12 *(Student page 129)* The only way to enclose P within a four-sided figure—the minimum—is to locate P at the center of the square. The region around P is then a square. Moving P pushes one corner of that region off the paper, while pulling the opposite corner onto the paper. We see that actually only two corners of the potential quadrilateral can get "clipped," so six sides is a maximum.

Problem 13 *(Student page 129)* Roughly speaking, moving P from the center towards a corner produces an enclosing region with six sides, and moving P from the center towards the midpoint of a side produces an enclosing region with five sides.

The exact boundaries of the areas can be determined by experiment. Or you can reason it out like this:

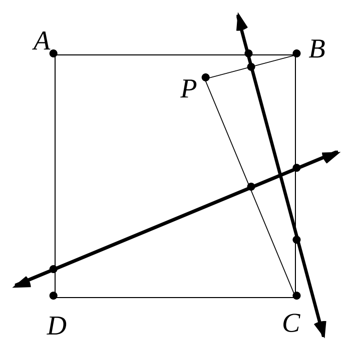
The idea behind this solution comes from Marvin Freedman, a mathematician at Boston University.

You get a new “clipped” side when two creases intersect *outside* the square:



Clipped side

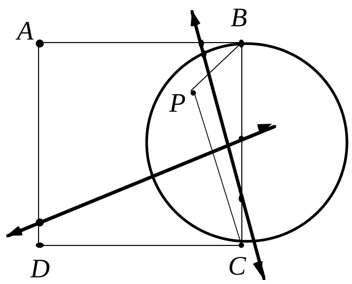
If the creases intersect inside the square, you get no new side:



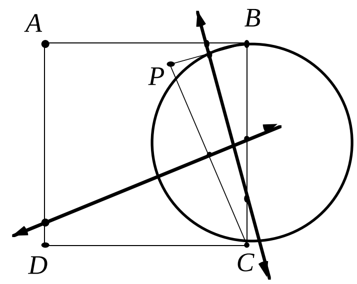
No clipped side

But look at $\triangle\ PBC$ in each of these pictures. The creases are precisely the perpendicular bisectors of sides $\overline{PB}$ and $\overline{PC}$, so you get a new side precisely when these perpendicular bisectors intersect outside the triangle. But *that* happens precisely when $\angle BPC$ is obtuse (why?).

We see that $\angle BPC$ is obtuse if and only if it lies inside the circle whose diameter is $\overline{BC}$:



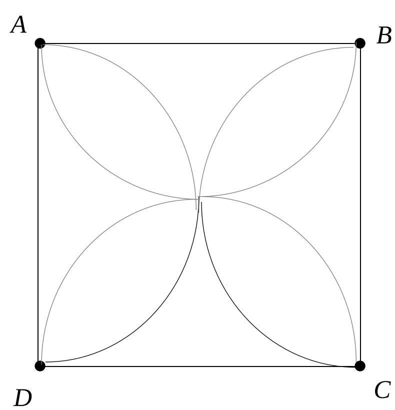
$\angle P$ *obtuse*



$\angle P$ *acute*

But it's the same story for every side of the square.

If you think about this a bit, you'll see that the picture below contains some interesting regions.



How many sides do you get if P is in a "leaf" of the rosette? What regions produce a P that leads to a 5-sided polygon?

Problem 14 *(Student page 130)* The software equivalent of creases are the perpendicular bisectors of the four segments connecting point P with the corners of a constructed square.

Circle Intersections

Problem 15 *(Student page 132)* When the circles intersect, $\triangle DGE$ has sides of length DE, $AC(= DG)$, and $BC(= EG)$. DE is fixed, but if $\overline{AC}$ gets too short, then $DG + DE$ will no longer be greater than GE, so no triangle is possible. Here is another way to say this: If one of the radii is too large, the smaller circle will lie completely inside the larger, and the circles will not intersect.

Problem 16 *(Student page 132)*

A *locus* is a set of points that uniquely satisfies a condition; in this case the condition is that $\frac{PA}{PB}$ is constant.

a. Moving A will, in general, make F and G travel along what appears to be a circle. Formally, the conjecture might be stated this way:

Locus: fixed ratio from two points The locus of points P, whose distance from D and E is at a fixed ratio, is a circle.

b. The radius of the circle depends on the position of C. If C is near A or B, the radius of the path of F and G is small. The nearer C is to the midpoint of $\overline{AB}$, the greater the radius of the circular path. If C is exactly *at* the midpoint of $\overline{AB}$, then we know that F and G travel along the perpendicular bisector of $\overline{DE}$.

c. The role that A and B play is identical: each stretches $\overline{AB}$, leaving C in a position that preserves the relative lengths of the two parts (the ratio) invariant. The effect on the circles is the same, whichever endpoint is moved.

Problem 17 *(Student page 132)* Moving C along $\overline{AB}$ makes F and G move in a path that appears to be an ellipse. Here is one definition of an ellipse:

DEFINITION

An *ellipse* is the set of points whose total distance from two fixed points (called the foci of the ellipse) is a constant.

Given this definition, you can show that this construction *does* generate an elliptical path.

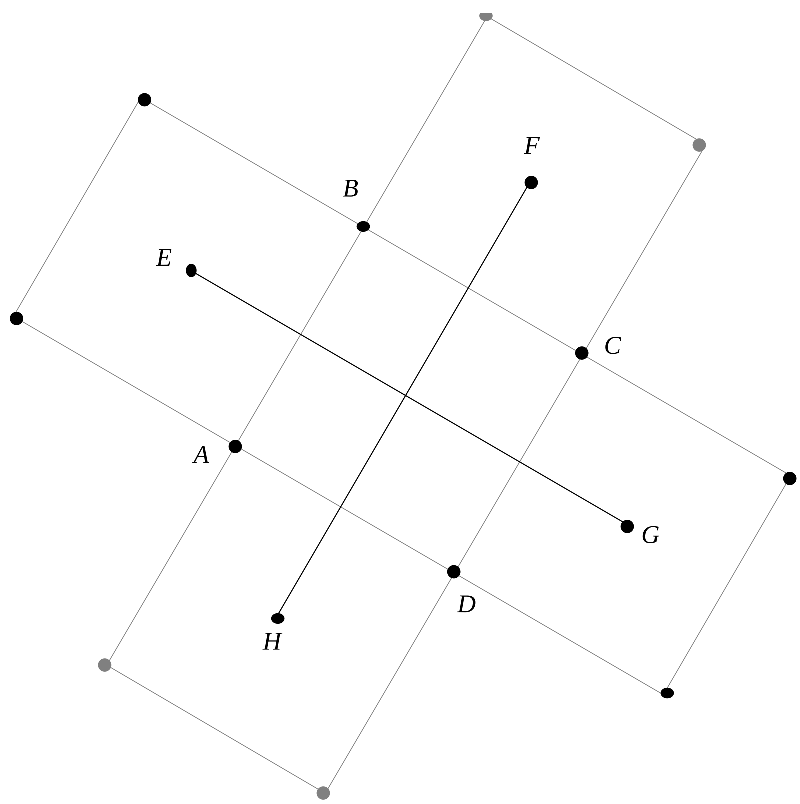
Centers of Squares

Problem 19 *(Student page 133)* Use the diagonals.

Problem 20 *(Student page 134)* The segments connecting the centers of the squares are perpendicular to each other and the same length.

Problem 21 *(Student page 134)* A complete proof that the segments are perpendicular and congruent is difficult; one such proof using complex numbers and rotations can be found in the *Teaching Notes*.

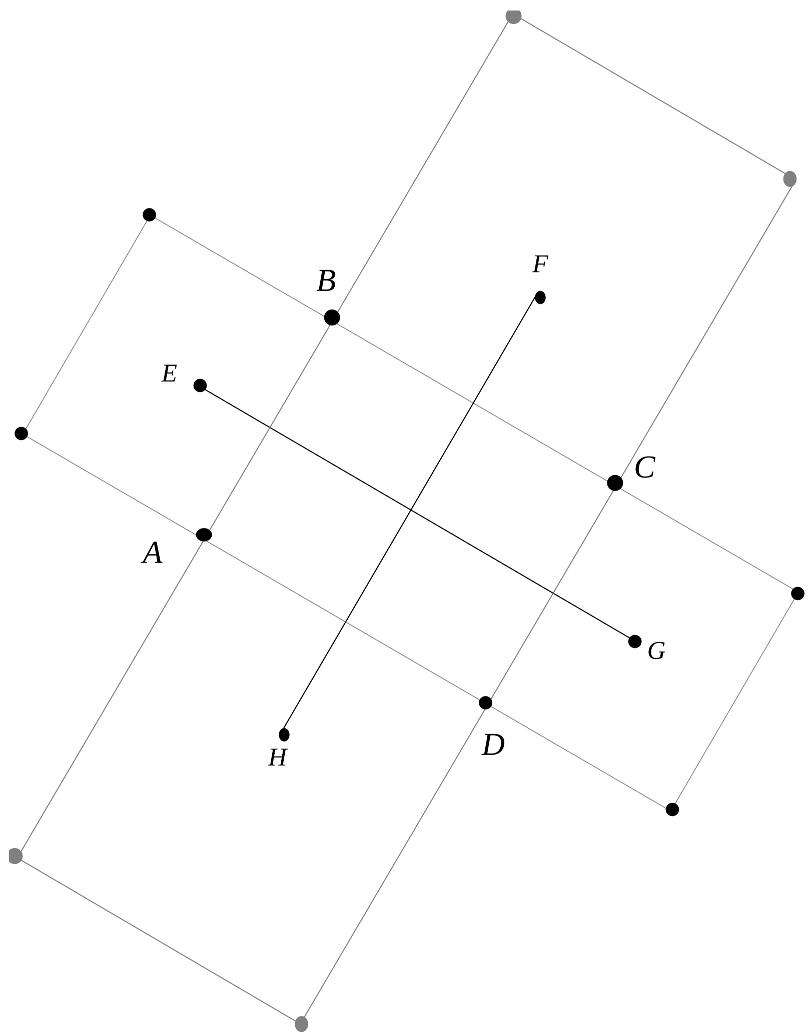
One way to look for geometric explanations is to consider special cases. For example, what if the original quadrilateral is a square?



Because all of the angles involved are 90 degrees, segments $\overline{EG}$ and $\overline{FH}$ are parallel to sides of the original square, and are therefore perpendicular to each other. Furthermore, we can find the lengths of these segments. If the original square has side s, $EG = FH = 2s$.

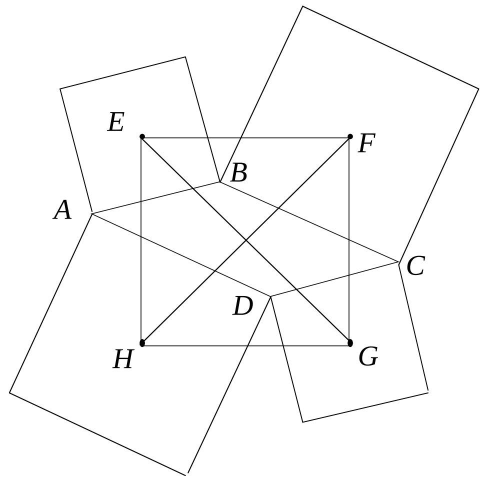
INVESTIGATIONS OF GEOMETRIC INVARIANTS
(continued)

Similarly, if the original quadrilateral is a rectangle, the segments will again be parallel to sides of the original, and perpendicular to each other.

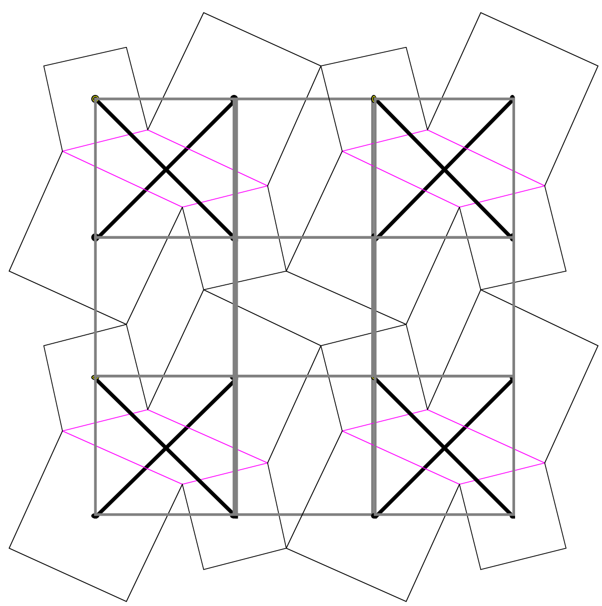


Again, we can deduce that the segments are congruent. If the dimensions of the original rectangle are $l \times w$, then $EG = FH = l + w$.

If the original quadrilateral is a parallelogram, things get more complicated. One thing to notice is that when $ABCD$ is a parallelogram, then $EFGH$ is a square.



There are many reasonable ways of proving this, but symmetry looks so promising that it is tempting to try first. Again, there are many reasonable ways of using symmetry. Here's a picture that suggests one way, but the details are left to you.



Problem 22 *(Student page 135)* If the original quadrilateral is a rectangle, the new one will be a square. Yes, the result can be a trapezoid but, remarkably, it appears

that only symmetric (isosceles) trapezoids can be made. It appears that if the outside quadrilateral has two pairs of parallel sides (is a parallelogram), it must be square.

Problem 23 *(Student page 135)* The area of the new quadrilateral appears always to be at least twice the area of the original quadrilateral. (But this depends also on what you consider a quadrilateral!)

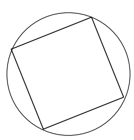
Constructing Invariants

Problem 24 *(Student page 136)* The only important thing to know is that the circumference is directly proportional to the diameter. To double a circle's circumference, one needs only to double its diameter.

Problem 25 *(Student page 136)* There are many different solutions. Here is a useful fact that will help you find "interesting" solutions: If the base of the triangle is fixed, the vertex can slide along a line parallel to that base without changing the area of the triangle.

Possibly the simplest solution to the problem is to divide a rectangle in half diagonally. This makes one side of the rectangle a base of the triangle, and places the third vertex of the triangle on a third corner of the rectangle. You can allow the shape of the triangle to vary more by placing its third vertex anywhere along the side of the rectangle opposite (and therefore parallel to) the triangle's base. In either case, the resulting triangle has exactly half the area of the rectangle.

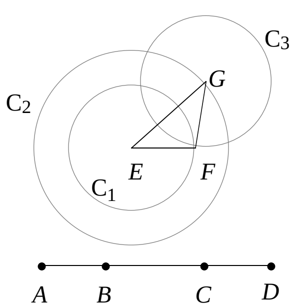
In this case, $\frac{\text{Area of circle}}{\text{Area of square}} = \frac{\pi}{2}$.



Problem 26 *(Student page 136)* Anything that keeps the side or diagonal of the square in proportion with the diameter or radius of the circle will guarantee that the ratio of the areas will not vary.

When comparing areas, the square whose side matches the radius of the circle has a particularly important relationship to the circle. Here is a solution that may take even fewer steps to construct: use the circle's diameter as a diagonal of the square.

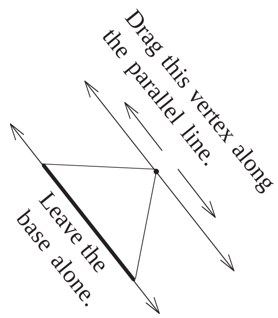
Problem 27 *(Student page 136)* The main task in this construction is to create three line segments whose total length is fixed. Once that is done, you can use methods you've used earlier to create a triangle from those sides (as long as their lengths do not violate the Triangle Inequality).



One partial approach to this problem uses a fixed segment ($\overline{AD}$, in the picture) to determine the perimeter of the triangle. Movable points B and C create subdivisions of that segment that determine the sides of the triangle. These subsegments change in length, but always sum to AD. (You must be careful to construct B and C in a way that does not allow them to slide past one another. Why does that matter?)

The variable segments along $\overline{AD}$ define the radii of the three circles, C_1, C_2, and C_3. In this setup, circle C_2 (radius BC) determines the length of $\overline{EG}$, and circle C_1 (radius AB) determines the length of $\overline{EF}$. Any point on C_2 can be used as a center for C_3 (radius CD), which completes the triangle.

Can you find a way of constraining the movement of B and C so that the lengths never violate the Triangle Inequality? Can you find a way of constructing this triangle so that the vertices of the triangle can, themselves, be moved while keeping the perimeter invariant?



Problem 28 *(Student page 136)* One way to solve this problem is to use what you know about chords from Problem 13 of Investigation 1.15.

Another way to keep the area of a triangle invariant is to construct two parallel lines and let the distance between them be the altitude of the triangle. A segment on one of the lines serves as the base. As long as the base is undisturbed, the parallel line guarantees a constant height, and therefore a constant area.

Problem 29 *(Student page 137)* To keep the perimeter of a rectangle constant, it is enough to keep the semiperimeter (half-perimeter) constant. Because this involves summing only two lengths, the same approach used in Problem 27 above can be used here, and will work even better than it did for the triangle.

Problem 30 *(Student page 137)* To keep the area of a rectangle constant, you must construct a pair of lengths that have a constant product. Again, you can use what you know about constant products from Problem 13 of Investigation 1.15.

Having constructed a pair of lengths, use circles centered at the intersection of two perpendicular lines to mark off one length along each perpendicular to form two sides of the rectangle.

GUESS-AND-CHECK

One reason people invent a formula or an *algorithm* is that it can take a long time to guess and check all the possibilities. Even so, guessing can still be a good place to start. Checking the guesses might lead you to notice a pattern that makes it unnecessary to check *every* possibility.

Problem 1 *(Student page 138)* The notes in the *Teaching Notes* and the *Solution Resource* for *Optimization: A Geometric Approach* suggest an approach using Logo procedures or spreadsheets to perform the calculations.

In this solution, we will focus on what you can learn by guessing and checking.

As an attempt at systematic guessing, we'll start with all jeans. What happens if you try to buy as many jeans as possible for your $250? At $29.95 each, you can buy 8 pairs of jeans for $239.60. Is this the best you can do? After all, it leaves only $10.40 leftover, not even enough to buy one shirt at $15.99.

Without some more checking, you can't tell if this is the best solution, but already you know that you need to check only eight more calculations—the change that is left when you purchase 7, 6, 5, . . . , or 0 jeans and however many shirts that this purchase allows.

If you buy just 7 pair of jeans, the remaining money is enough to buy 2 shirts with $8.37 leftover. That's an improvement. Only seven more possibilities to check.

If you check them all, you'll see that the solutions do *not* follow a perfectly neat pattern, improving gradually until they reach the best and then gradually getting worse and worse again. If there were a nice pattern like getting better and then getting worse, the last one before the change would be best, and you wouldn't have to check the rest of them.

Problem 2 *(Student page 140)* One extremely efficient algorithm is to guess (0, 0) first and move halfway in whatever direction your partner's directions lead you, guess again, and again move only halfway according to your partner's directions, and so on.

For example, if you guess (0, 0) and your partner says "south and west," you will move to $(-4, -4)$ (the middle of the third quadrant) for your next guess, and eliminate everything north or east of (0, 0) from all further consideration. If the next clue is "east," you will move to $(-2, -4)$. Good luck could, of course, get you to the homework on the first guess, but this algorithm *guarantees* that even with the worst luck you will never need more than four guesses (on a ± 7-unit grid) to know where the homework is located.

Problems 4–5 *(Student pages 141–142)* This is a perfect example of the guess-AND-CHECK strategy at work. There is only one spot where the treasure could be, and fancy techniques for figuring out that spot are no more efficient in this problem than guessing a couple of times and seeing what happens. (Some starting spots force

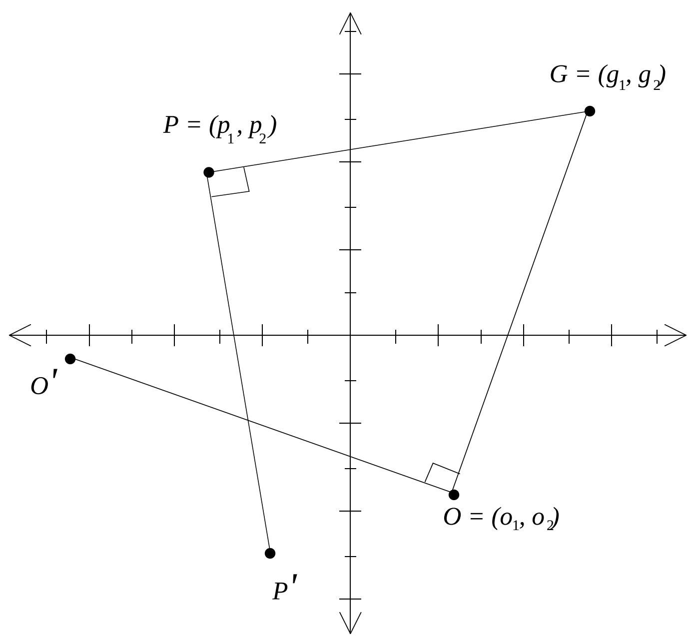
Captain Bonny to wade through the shark-infested waters on her way to to the burying place!)

Two shrewd guesses (and careful following of directions) will show you how the hiding place depends on the starting point you chose (or rather, how it doesn't). From there on, it should be easy.

The authors don't expect students in the class at this point to come up with this reasoning unless they have had significant experience with coordinates in the past.

One way to *prove* this surprising result is to put the whole system down on a coordinate plane and look at what's happening. We'll put the gallows at the point $G = (g_1, g_2)$, the palm tree at $P = (p_1, p_2)$, and the oak tree at $O = (o_1, o_2)$.

The easiest solution involves putting the gallows at the origin and one of the trees along one axis, but this solution is often more convincing that the location of the gallows drops out of the calculation.



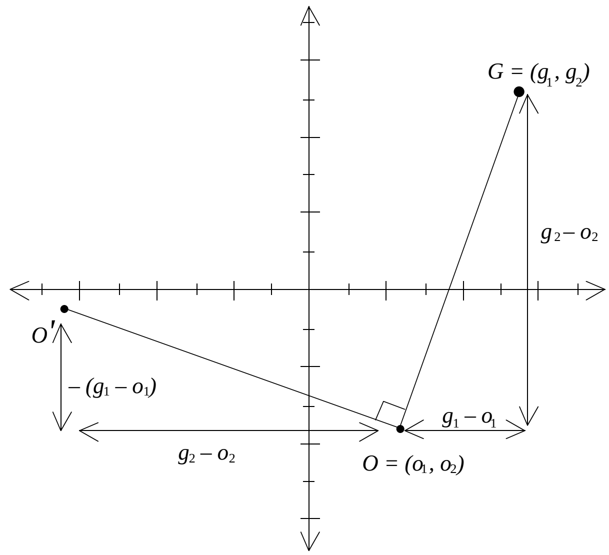
We want to calculate the coordinates of the points O' and P' in the picture above. Using a few facts about slope and distance, we can find them:

$$O' = (o_1 + g_2 - o_2, o_2 - g_1 + o_1)$$

$$P' = (p_1 - g_2 + p_2, p_2 + g_1 - p_1).$$

We don't even need to resort to formulas here. For the oak tree, we want to go over the same amount we went up and up the negative of the amount we went over. For

the palm, we want to go over the negative of the amount we went up and up the same amount we went over.



Using the midpoint formula to find the midpoint of OP', we calculate

$$M = \frac{1}{2}(O' + P') = \frac{1}{2}(o_1 - o_2 + p_1 + p_2, o_1 + o_2 - p_1 + p_2).$$

The formula for M does not depend at all on the point $G = (g_1, g_2)$ (though the points O' and P' do depend on G).

For you to prove: Let N be the midpoint of $\overline{OP}$ and M is the midpoint of $\overline{O'P'}$ in the original picture. Show that the distances are equal: $NM = NA = NB$.

You won't need it for *this* problem alone, but setting up the equations can be very instructive! Combining the two equations leads to $a(22 - a) = 21$, showing *non*-integral solutions along with the integral one.

Problem 6 *(Student page 142)* You *could* set up the equations

$$ab = 21 \quad \text{and} \quad a + b = 22$$

and try to solve this system. Just for the sake of *this* problem, that would be quite a waste! There are only two possibilities: 7 and 3, or 21 and 1. Only one combination works.

Problem 7 *(Student page 142)* Here is a guess to check: if the pencil costs \$2.00 and the pen costs \$2.89, the total cost is \$4.89. That's only off by 10 cents, which can be split between the two items. The pencil costs \$2.05; the pen costs \$2.94.

Solutions

Investigation **1.20**

Student Pages 143–148

REASONING BY CONTINUITY

Problem 1 *(Student page 143)* The temperature *surely* was 71.5° at *some* time, because, in rising from 64° to 86°, it cannot "skip over" any temperature in between. On the other hand, you cannot be sure *when* it was 71.5°, because the temperature may not have changed at a constant rate.

The formal statement of this theorem requires precise definitions of "continuous" and "something is changing." Those definitions are usually given in terms of *functions* and *limits*. For now, these ideas must remain intuitive, although you'll have a brush with the idea of limits in Problem 3 below.

This assumption about continuous change is a basic theorem of calculus, and, although it isn't proved here, it is such an important idea that we will state it informally as a theorem anyway.

THEOREM ***Intermediate Value Theorem***

If a quantity is changing in a continuous way between two values a and b, then it must pass at least once through every value between a and b.

Problem 2 *(Student page 143)* As with the previous problem, there is no information that would allow you to say *when* the population hit, 10,000 but, in this case, you cannot even be sure that it *does* hit 10,000 exactly. The population of a town *can* make sudden jumps, skipping over intermediate values. For example, if the population is 9,996 and a family of eight moves into town at the same time, the population will jump to 10,004. Population change is *discrete*, not *continuous*, the way temperature change is.

Problem 3 *(Student page 143)* This problem feels very much the same as the temperature problem. We understand speed of a car to change in a continuous way; when a car accelerates from 0 to 60, it can't skip over intermediate speeds. In particular, it must, at some time, pass through 32 mph.

That's the *answer*, but there's still a problem! Speed *depends* on time: it is a measure of how much distance is traveled over a given period of time.

Suppose that the car's speed is constantly changing. The car never lingers at any of the intermediate speeds but continuously accelerates from 0 to 60, always increasing its speed until 60 mph is reached. Is there "a time" when the car is going 32? That is as much a matter of philosophy as it is a matter of mathematics.

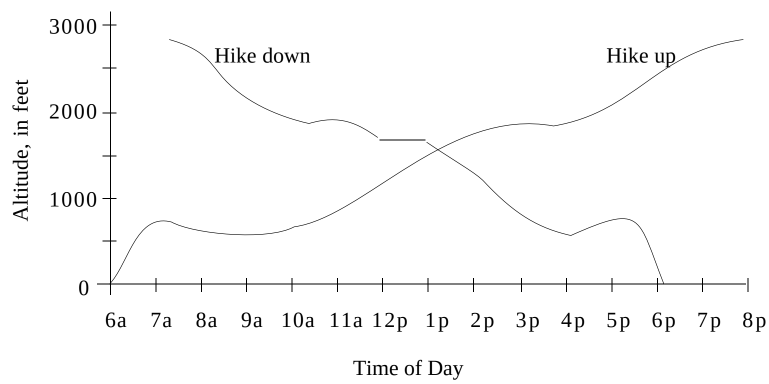
If there *is* such a time, how long is it? Well, it is of *no length at all*, because the problem already stated that the car's speed never stays the same, not even for an instant! Each microsecond, it is greater than it was the instant before. So even if

it seems comfortable to say that there was a single point in time at which the speed "passed through" 32, it is hard to know what "speed" even means at a single point in time! A point in time has no duration. If no time elapses (the "period" of time that has no length at all) then the car is not moving at all! So how can it have *any* speed?

In one form or another, this paradox has worried mathematicians and philosophers for *thousands* of years. Calculus is the branch of mathematics that provides a way to think about such problems. It looks at *limits*. Over short periods of time, the car's speed changes only a little. Over shorter periods of time, it changes even less. During *one* very narrow time period, the car is going roughly 32 mph. We can find a narrow period in which it is going roughly 32, and squeeze in on both ends of that period to find a "spot" where we agree that the only sensible speed to assign to that point in time is 32 mph. It *feels sensible* to call the speed *at* this point 32.

The problem is that we cannot make any sense out of the usual computation that defines speed because, in this case, that would require us to divide 0 distance by 0 time. So, in order to make sense again, we agree to *extend the definition* so that we can talk about a speed at a single point, which physicists call "instantaneous velocity."

Problem 6 *(Student page 144)* One way to clarify the situation is to graph altitude vs. time for both days.

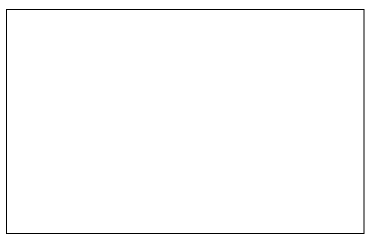


Notice that the starting and ending *times* of the two trips (up and down) don't have to match up (and don't), but that if you are traveling along the same path, the starting and ending *altitudes* do have to match up: the trip down starts at the same altitude that the trip up ended. (Otherwise, there'd be a lot to explain!) Whatever rises and dips the path takes on the way up are still there on the way down again, but they come in reverse order and may be compressed or expanded depending on the amount of time it takes you to pass by them. The "brief stop in the middle to catch your breath" on the way down appears as a level stretch in the graph from about noon to 1. (It was not

Can there be more than one crossing on the graph on the previous page? Could there be exactly two crossings?

such a brief stop after all! Lunch?) Even with all those changes, the two lines *must* cross somewhere. Where they cross, you were at the same altitude, at the same time of day. Another way to think about this problem: Instead of considering one person making two trips on different days, consider (an equivalent problem) two people — one climbing and one descending — making trips on the same day. If they both leave in the morning (not necessarily the same time) and arrive in the evening, and they take the same path, they will surely pass each other during the day.

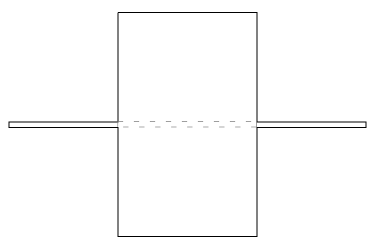
Problem 7 *(Student page 145)* Small cutouts give very little “side” to the box, and such a shallow box has very little volume. Slightly larger cutouts will increase the volume. At some point, however, the volume begins to decrease again. If the cutouts are too large, the folded-up container will be “deep” but so narrow that it is really more like an envelope than a box and again has little volume.



No cutouts, no box, no volume

Tiny cutouts, shallow box, tiny volume

Somewhere in between a small cutout and a large cutout . . .



Large cutouts, deep envelope, tiny volume

Problem 8 *(Student page 145)*

a. If the volume of the box is small for small cutouts, small for large cutouts, but larger somewhere in between, then there must be *at least one* place where the volume is the largest. (Of course, by experimenting alone, we cannot be sure that the volume doesn’t do strange things in between, such as

rising and falling a few times before it begins to settle back down to the low volumes we predict when the cutouts are large. But that doesn't matter. That only says that there might be *more than one* place where the volume is tied for highest.)

b. The answer depends on how you interpret "a box with the smallest volume" and how "practical" you want to be!

On the one hand, zero is the smallest volume a box can have, and you can already see how to create a "box" with zero volume. On the other hand, the "box with zero volume" is not a box: you cannot produce any kind of box at all (that is, anything with a volume) unless the cut-out must be strictly *between* (not including) 0″ and 2.5″. Something close to either of these endpoints gets a real box (of sorts), but something even closer would get an even less spacious box, and something even closer than that would get an even *less* spacious box, and so on. So, of the things that are actually boxes at all, no one of them is smallest, because you can always get a little smaller.

And on the third hand (strange anatomy!), if we're talking about real boxes made out of real 5″ × 8″ cards, we have to consider the "real world," in which you can't assume continuity (how would you fold an atom?), and so you can't "always get a little smaller." Long before we have to imagine folding atoms, we've reached the limitations of what's possible to do with thick paper.

Problems 9–10 *(Student page 145)* One way to test the volumes of various boxes is to fill them with something like sand and then measure the sand (using measuring cups or weighing it). If you start with a 5" x 8" card, a cutout of 2" produces the maximum volume. It takes calculus to prove this, but tabulating data, graphing an equation, and playing with a sketch are all ways you could conclude that the value should be about 2".

Problem 11 *(Student page 146)* To the eye, the figures appear to be mirror-symmetrical. Cutting along the line of symmetry divides these figures into two matching pieces. Pieces that fully match must have equal area.

Problem 12 *(Student page 146)*

a. If the three top shapes *are* what they *appear* to be, then each can be cut into two congruent parts.

b. For the bottom three figures, it doesn't seem as though *any* straight line will cut any of them into congruent parts. Even so, it does look like they could

be cut into two portions that have equal area (though different shape). How can you *know* if you've succeeded? You can't, except in the world of ideas. And you can know only by making measurements, which doesn't look easy at all! For the rest of these problems, we will stick to the world of ideas in which shapes are generally continuous.

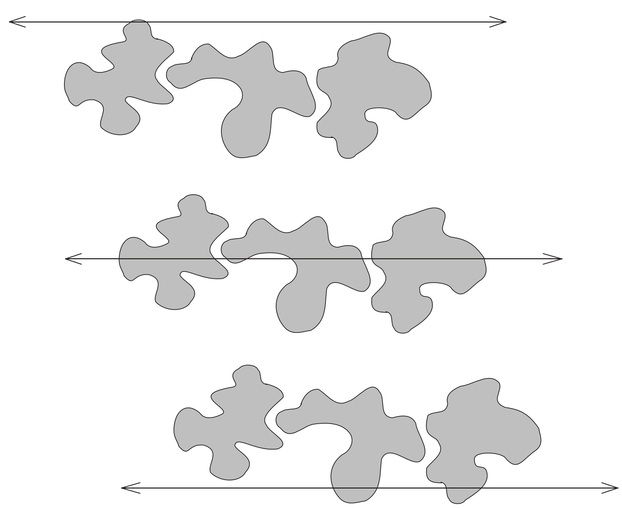
Problem 13 *(Student page 146)* It will probably be frustrating trying to find a shape (Problem 13) that cannot be cut in two portions of equal area by any straight line or by only *one* particular straight line and no other. In fact, Problem 14 asks you to complete a line of reasoning begun in the Student Module that shows that there must *always* be some line that can cut a region in half, and can be extended to show, in fact, that there are infinitely many such lines for any region.

This argument is one more instance of the Intermediate Value Theorem stated earlier.

Problem 14 *(Student page 147)* If the line can be moved gradually from where it cuts off a little to where it cuts off a lot, then somewhere in between, it cuts off just half. Such a process of "slicing until one reaches the middle" can be started at any side of the blob and in any direction, so there are infinitely many ways to slice the area in half with a straight line. Finding the line is another matter! Knowing that there *is* one gives you no help in finding it. For irregular shapes, you may have nothing to go on except measurement, which is not easy at all and is never a precise matter. You can, however, find an approximate location for the line experimentally. One way to do this is to cut out the shape and find a way to balance it on top of a prism.

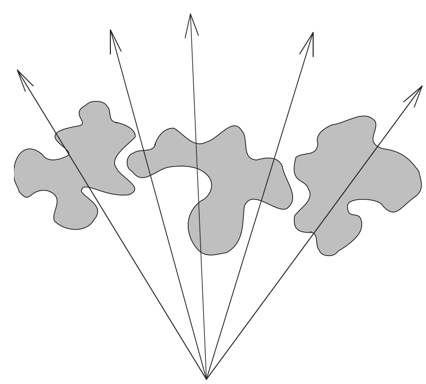
***Bisect* means to cut or divide into two equal parts.**

Problem 15 *(Student page 147)* The pictures below show that a line can be positioned so that it cuts off very little, about half, or almost all of the area of the three blobs. Somewhere along the way, it must bisect the area (cut off exactly half the area).

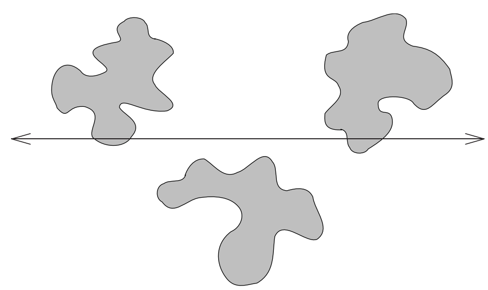


Although the previous figure shows only one orientation for the line, it could also cut *the combined area* in half if it were vertical or at any other slope. The same reasoning applies. Slide the line (at whatever angle it is) from a position completely on one side of the blobs to a position completely on the other side. Somewhere in the course of this process, it must cut the (combined) area exactly in half.

You could equally well imagine a ray of light sweeping across this three-island nation from a lighthouse on a rock. At one point, it certainly divides the (combined) area of the nation exactly in half, no matter where that rock is placed.



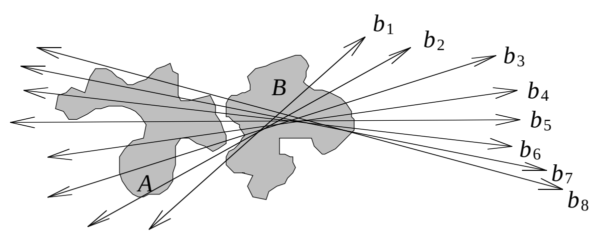
Problem 16 *(Student page 148)* If the three shapes are offset as they are here, then a line cannot pass through all of them simultaneously. Therefore, while one line can still bisect their combined area, there are situations in which one line cannot bisect each shape individually.



Problem 17 *(Student page 148)* If there are only two arbitrary regions, it is possible not only to find a line that bisects their combined area, but it is also always possible to find a single straight line that will bisect each area individually. Many

people quickly convince themselves that this is *probably* true, but showing that it *is* true (and showing *why* it is true) is really very challenging. (The authors discussed this a long time before settling on a solution they all believed! It took even longer to find a "simple" solution they all believed!)

Call the two regions A and B. We know already that at any slope at all, there is a line that bisects either region's area. So let's focus on B. Imagine a continuum of such lines. The picture of b_{slope1}, b_{slope2}, b_{slope3}, and so on is certainly not continuous (the change in slope between successive bisectors is pretty large), but it gives the idea.



As you can easily see, most of the bisectors of B's area certainly do not bisect A's area. But all we need to do is prove that at least one of them *does*.

Starting with b_{slope1}, we see that *all* of A's area is above it. By b_{slope8}, almost all of A's area is below it. Now, we cannot say that all of the little $b_{slope.i}$s are concurrent at a single point, and so we cannot say for sure that as they sweep across A the part of A's area under the little $b_{slope.i}$s is always increasing. But that doesn't matter. Whatever rising and falling may take place in the middle, somehow the part of A's area below B's bisector grows from none to all, and so somewhere it must be just half. We've got it!

Problem 18 *(Student page 148)* There *is* a plane that cuts *all three parts* of such a sandwich exactly in half! Most people find the pictures harder to make in their heads, but the reasoning is very much the same.

The (*hard*) first step is to find a solution to a simplified version of the problem. You must convince yourself that (1) there is a plane that bisects each of *two* parts of the sandwich, and (2) it is possible to continuously tilt the plane in space in a way that keeps it bisecting the two parts. (If you can do this, you have a 3D version of the lines you found in Problem 17 that would cut one region in half and could be tilted.)

With that part done, the rest is easy. Since the plane can tilt so that it effectively cuts off very little, some, or most of the third sandwich part, there must be a position that *bisects* the third sandwich part.

Definitions and Systems

Problem 1 *(Student page 150)* First, we need to decide what we mean by "straight."

If the two straight roads are straight like rays of light, or straight the way we mean it in a plane, then they won't stay on the Earth. We need to give a meaning to "straightness on a sphere." But, in any event, two roads aiming due north from the equator will eventually reach the North Pole (that's what it means to be going north!). There, of course, they will meet. So, no matter how far apart they were at the equator, they keep getting closer to each other as they approach the North Pole.

Problem 2 *(Student page 150)* In the plane, there is a unique (only one) shortest path between any two points: we call that a (straight) line segment. If we extend this segment in both directions we get a (straight) line, which is infinite.

Problem 3 *(Student page 150)*

a. The shortest path between any two points on a sphere is an arc of a "great circle"—a circle around the sphere whose diameter is the same as the diameter of the sphere. (One might still reasonably want to call these shortest-path arcs *segments*, just as one might want to call the great circles *lines*.) Great circles, which are as big as the equator of the sphere, are thus the "straight" paths (or *geodesics*) on spheres.

b. In the plane, the shortest path between any two points is unique. But imagine a pair of points at opposite ends of a diameter of the sphere. (The North and South Poles are a single example of such a pair of points.) *Many* paths between these two points are equally good ways to travel the shortest distance between them! To get around this messiness, a geometry defined on the sphere often regards such pairs of (Euclidean) points like these as single (spherical) Points. Then, through any two such (spherical) Points, there is again just one line.

Problem 4 *(Student page 151)* How do you define a triangle? If all you require is three segments that connect at their endpoints to enclose a region, then triangles can be drawn on a sphere, because segments (arcs of great circles) can be drawn on a sphere.

Problem 5 *(Student page 151)* How do you define a square? This is trickier because there are several possibilities that are equivalent on a plane, but *not* equivalent on a sphere.

- *Definition 1: A square is a regular four-sided polygon (a quadrilateral with four*

straight, matching sides and four matching angles). Regular polygons (figures with congruent sides and congruent angles) can be drawn on a sphere. (The four angles of the regular quadrilateral won't be 90°, but this definition said nothing about right angles!)

- *Definition 2: A square is a quadrilateral with four congruent sides and four right angles.* You can construct a right angle by using, for example, the equator and a longitude line (circle through the North and South Poles). *Four* right angles, however, are impossible. Some experimenting shows that after three right angles have been built into a quadrilateral, the fourth angle must be obtuse.
- *Definition 3: A square is a quadrilateral with four congruent sides and a right angle.* This is possible. Create a pair of perpendicular sides and mark off equal distances along both. Through each endpoint, great circles may be passed at any angle, so they can be adjusted to produce two other sides that are the same length as the first two. (The angles will not all be congruent, but the definition said nothing about congruent angles!)
- *Definition 4: A square is a rectangle (two pairs of parallel sides and a right angle) with congruent sides.* Forget it! Parallels are not possible on a sphere. So parallelograms, and therefore rectangles, and thus, by *this* definition, squares, are impossible.

So, *can* A Square exist on a sphere? It depends on what you mean by "A Square."

Problems 6–7 *(Student page 151)* A triangle with three right angles can be constructed by creating a right angle at the North Pole and using the equator for the third side. In fact, the sum of the angles of any triangle on a sphere is not an invariant as it is in a plane. The sum will always be greater than 180° and less than 540°. Amazingly, the angle sum has a simple relationship to the area of a triangle (when that area is expressed as a fraction of the area of the sphere)!

Problem 8 *(Student page 152)* If we do not specify that the lines must be in the same plane, we can find an infinite number of lines that are neither parallel to, nor intersecting with, a given line. One way to picture this: choose a pair of lines that intersects in a plane and "lift" one or them "out of the plane." The lines become nonintersecting without becoming parallel. These are called skew lines.

DEFINITION

Skew lines are lines in parallel planes that are not parallel to each other.

Problem 10 *(Student page 152)* On a plane, perpendicularity and parallelism are tied closely together. If two distinct lines l and m are both perpendicular to line k, then lines l and m must be parallel to each other. But there are no parallels on a sphere, so how do you want to define perpendicularity for a sphere?

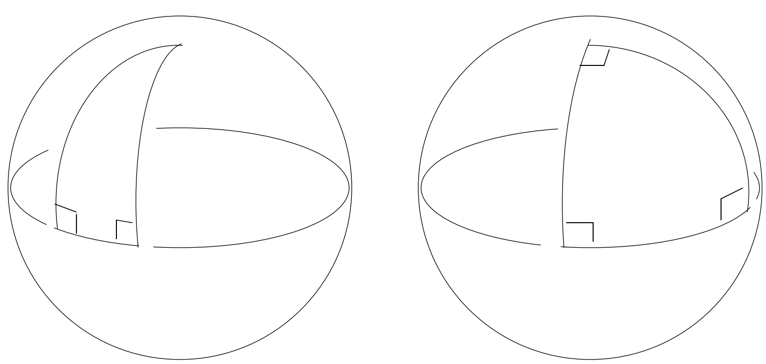
There are at least two possible ways you might think about this. You might start by *defining* the properties of "perpendicular" in terms of the properties of parallels (as might be tried on a plane). In that case, since there are no parallels on a sphere, there might also not be perpendiculars.

But that's very restrictive. And, in fact, the relationship in a plane between perpendicular and parallel—if l and m are both perpendicular to k, then l and m must be parallel to each other—is not even true in three dimensions (lines l and m *might* be parallel, but they might also be skew).

Another way to define perpendicular is just "intersecting at a 90° angle." Conventionally, angle is measured *locally*, right *at* an intersection, so it does not depend on what happens somewhere else on the surface. Perpendiculars exist on a sphere because, for example, the angles between any longitude lines and the equator are right angles (measured *at* the intersection with the equator). But, far away from where perpendicularity was measured, all longitude lines intersect, so they are not parallels.

Problem 12 *(Student page 152)* Here are some experimental results that suggest parts of the relationship between the area of a spherical triangle and the sum of its angles. The rest of the details are left for you to work out.

A spherical triangle with three right angles covers $\frac{1}{8}$ the area of the sphere. A spherical triangle with two right angles and a third angle of measure $\frac{360}{n}^\circ$ appears to cover $\frac{1}{2n}$ the area of the sphere.

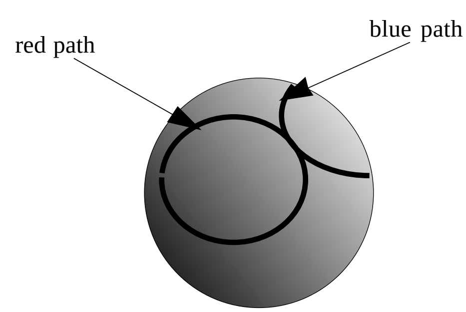


The smaller the third angle, the closer the angle sum to 180° and the smaller the area. When the angle sum is very near 180°, the area is very near zero (compared to the area of the sphere). Similarly, any triangle that is very tiny compared to the sphere will "feel" as if it is on a plane (just as triangles we draw on the Earth are not large enough to experience much of the bend of the sphere they're on). The less area inside the triangle, the less affected its angles will be by the curvature of the Earth and thus the closer their sum will be to 180°.

Problem 13 *(Student page 153)* No matter how long and wiggly each of the two loops is, the two must always intersect in at least two points: the starting place and somewhere along the route. But they do not have to be of the same length. The blue could simply be a circle of a smaller radius than the red.

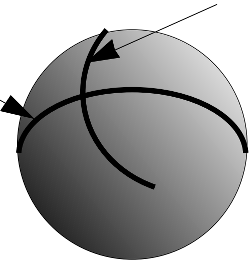
Problems 14–15 *(Student page 153)* On a sphere, as long as one of the two journeys was not along a straight line (great circle), then it is possible that is was shorter.

It is also possible on a sphere—provided that at least one of the two journeys was not along a straight line—to create some kind of blue path that returns home without meeting the red path en route.



According to the story, however, the paths are not merely tangent, as shown in the picture above, but they actually *cross* each other at Flatsburgh, "a red one starting east and returning from the west and a blue one starting north and returning from the south." On a sphere, a closed loop like the red path will divide the sphere into two regions. Think of this east-west path as a boundary line between the two regions, a

region north of the loop and a region south of it. (It may even help to think of coloring the entire northern region yellow and the southern region green.)



The blue path starts into the north (yellow) region but returns from the south (green) region. The blue path cannot get from the yellow region to the green region without crossing the red boundary along the way, but the story also says that that is exactly what happened!

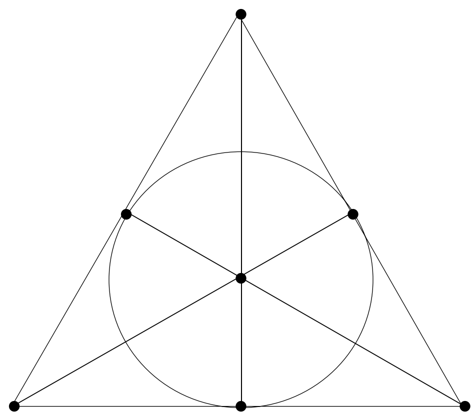
How can this conflict be resolved? Unless the story is untrue (or the red thread was broken or buried or hidden), the only other way to resolve the problem is to assume that the world *cannot be a sphere*.

Especially if A Square traveled straight both times, we must consider other possible shapes for his world. One possibility is that A Square's world is a torus. Then the east-west (red) path might circle around the periphery of the torus, while the north-south (blue) path might loop through the hole. The blue path would be shorter and would not meet the red path along the way.

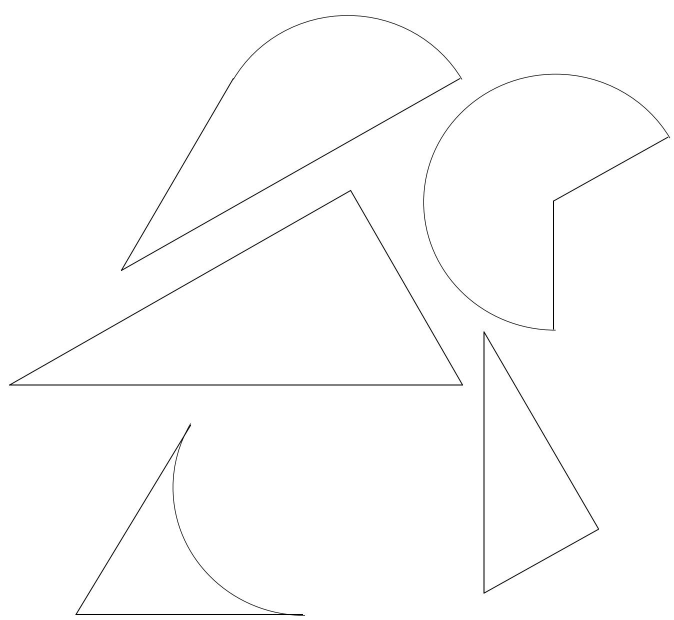
Using the same reasoning, A Square's world could be a two-holed torus, or a three-holed torus, or . . . .

Problems 16–19 *(Student pages 153–154)* "Models" that follow the four rules could *look* quite different from one another but must share a common *structure*. For one thing, if they follow all four of these rules, they will have exactly seven wires and

seven beads—no more and no fewer. Here's one "model" that follows the rules, using one curved wire and six straight wires.



Here are some triangles that exist in this seven-point geometry:



Are there quadrilaterals? Squares? Midpoints?

Problem 20 *(Student page 155)* The definitions given in the problem do not make it completely clear what a *segment* and its *length* are, because they do not explain the

word "between" or what is meant by "the shortest route." For example, the first column of the first block shows that there is a segment whose endpoints are A and P. But are "the points between them" F and K, or should we "wrap around" the other way to go "the shortest route" and include just U? That is, is the segment $\overline{AFKP}$ or $\overline{AUP}$?

Because the definitions don't make that clear, you can decide for yourself! And what criteria would be useful in making this decision? Consider how "interesting" or "rich" the geometry becomes, or how well it accords with other geometries that make sense to you, or how messy or clean the definition makes things.

For the following solutions, we've chosen to interpret "line" to be infinite (no endpoint), and therefore we allow segments to "wrap around." So, for these solutions, the segment whose endpoints are A and P is $\overline{AUP}$. We didn't seriously investigate the other possibility, so we don't know if it is better or not.

With the definition we've chosen, there are only four different segment lengths: (1, *row*), (2, *row*), (1, *col*), and (2, *col*). The length of $\overline{AUP}$ is (2, *col*).

Problem 21 *(Student page 155)* Here are three segments that meet at their endpoints and conform to the usual definition of a triangle: $\overline{AUP}$, $\overline{PT}$, $\overline{TWA}$. Here's another: $\overline{AF}$, $\overline{FR}$, $\overline{RA}$. The second triangle is *equilateral*! But there might be other sensible definitions of a triangle.

Remember the problem with drawing squares on a sphere? They *could* be drawn or *couldn't* be drawn, depending on how you defined "square." You will have the same problems thinking about rectangles in this 25-point geometry.

Problem 22 *(Student page 155)* In the plane, *all* the points that are equidistant from both ends of a segment are on the perpendicular bisector of the segment. Of those points, the one closest to both ends is also on the segment, so the two definitions pick the same point.

You may have already thought about what you might mean by a segment on a sphere, but this module has not discussed it. Given points A and B on a sphere and one line (great circle) passing through them, there are still *two* great-circle arcs whose endpoints are A and B. Are they two different segments (because they are both pieces of a straight line)? Or is a segment the shortest path between two points, in which case only one of them is a segment? If $\overline{AB}$ *can* be longer than a "half-line" (half a great circle), then you might imagine it running, for example, most of the way around the sphere. Then it is easy to see that the point closest to A and B and equidistant from them is in the gap between A and B and not on $\overline{AB}$ itself. In other words, the midpoint

of the segment is not on the segment! That's awkward, so perhaps we should treat a segment as we do on a plane: the shortest distance between two points. Then the two definitions of midpoint would come out the same again. (By the way, this messiness is cleared up even more by rethinking what is meant by "point" on a sphere.)

In the 25-point geometry, we have the same kind of problem, only it is not so clear that there is *any* way to make things completely neat. If we are just looking for the point at the shortest equal distance from two endpoints, the midpoint of $\overline{AU}$ would be K, a point that is equidistant from A and U, but K is not on the segment! If we insist that the midpoint be *on* the segment, then $\overline{AU}$—along with all (1, *row*) and (1, *col*) segments—has no midpoint at all. Of the two possibilities—segments with midpoints that are not on them or segments with no midpoints at all—neither feels familiar, and both seem awkward, so the decision should rest with which possibility enables you to do more things. Having the midpoints is probably more useful, but you'll have to investigate this one yourself!

NON-EUCLIDEAN GEOMETRIES

Problem 1 *(Student page 159)* It is quite popular to report that people did not know for sure until the 16th century that the Earth is spherical and not a plane. In fact, that is almost certainly a myth. Navigators looked at the stars and knew that different stars were visible in the southern hemisphere than in the northern, and they knew about the North Star and the rotation of the Earth. People's ideas about what rotated around what were certainly wrong for centuries—it *was* a real blow to discover that the Earth was not the center—but people who navigated knew *way* back that the Earth was not a plane.

On the other hand, not everybody traveled such long distances and, locally, it *was* easy to be fooled by appearances: though the Earth is spherical, it is so large that its local geometry looks and behaves as the geometry of a plane. It is like taking a ball and looking only at a small piece of its surface; it looks and behaves like a piece of plane. And whether one is "fooled" or not, most projects that required "practical" geometry—building a building or bridge, measuring a field, designing things to fit together, making a picture that looked right—were things in which the curvature of the Earth was entirely or almost entirely irrelevant. Plane geometry "ruled." (Pardon the pun.)

Problem 2 *(Student page 159)* In general, airplanes do travel along the arcs of great circles when they make long trips, for example, from Europe to the United States. But when it comes to short trips (for example, between Boston and New York), the difference between the shortest trip (along a great circle) and one that deviates slightly may be so small that other factors become more important. These can include reducing air-traffic congestion or noise near big cities, avoiding flying over hostile territories, dodging bad atmospheric conditions, and so on.

Solutions

Investigation **1.23**

Student Pages 160–162

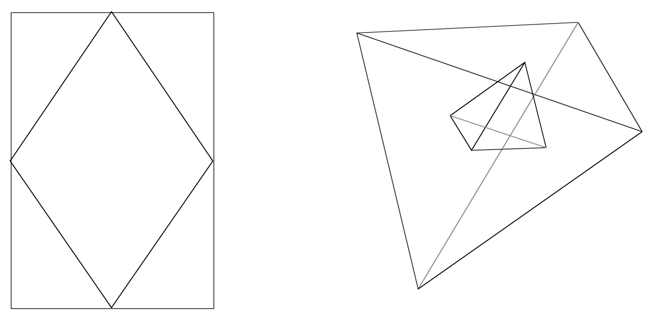
VISUALIZATION EXERCISES

For Discussion *(Student page 160)* The number 41 is special in this formula. If $n = 41$, the whole expression turns into $41^2 + 41 + 41$, which is certainly divisible by 41.

Problem 1 *(Student page 162)* If the gymnasium is rectangular, all midpoint-to-midpoint distances would be equal, so the path would be a rhombus.

Problem 2 *(Student page 162)* These four points are the vertices of a smaller tetrahedron.

Problem 3 *(Student page 162)* The sketches might look like this:



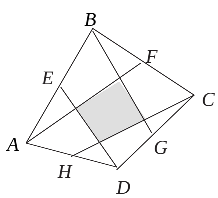
Can you picture each of these in your mind?

MIDPOINTS IN QUADRILATERALS

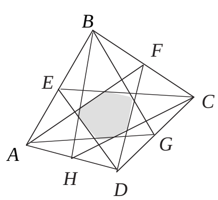
Problem 1 *(Student page 163)* To describe these constructions (especially the first one) without pictures, you must somehow be able to clarify *which* midpoints are connected to *which* vertices. One possible strategy is to use words like "clockwise" that impose an *order* on the sides. For example, you might describe the first construction this way:

> Construct four new lines by connecting each vertex of the quadrilateral to the midpoint of the "next" side (taking the sides in clockwise order). Shade the internal quadrilateral that is bounded by these four new lines.

Problem 2 *(Student page 164)* There are many things to find. We will mention two *conjectures* (no guarantees about their truth!), because (1) they are so different from the invariants you may be accustomed to looking for, and (2) they may give you ideas about searching for your own invariants.



Area ratio "often" 5:1



Area ratio 6:1 or greater

- In the first construction, the ratio of the area of *ABCD* to the shaded quadrilateral is "generally" 5:1, but it is not *always* that value. That is, the area ratio is an invariant over some changes in the shape of quadrilateral *ABCD* but not invariant over all changes in its shape. (A stronger conjecture would state exactly *what* shape changes were permitted!) But two complete invariants can be conjectured: (1) If the inner figure exists, it is always a quadrilateral. (2) The area ratio is never *less* than 5:1.

- The area of the shaded figure in the third construction is not invariant, but is never more than $\frac{1}{6}$ the area of the outer quadrilateral. It seems to reach its maximum, $\frac{1}{6}$, when quadrilateral *ABCD* is a parallelogram.

Problem 5 *(Student page 165)* You may want to look at the following characteristics of the quadrilateral:

- Are the opposite sides congruent?
- Are the opposite sides parallel?
- Do the diagonals bisect each other?
- Are opposite angles congruent?
- Are adjacent angles supplementary?

Problem 6 *(Student page 165)* You will explore the answer to this question further later in this section of the module.

Solutions

Investigation 1.25

Student Pages 166–169

WHAT DO YOU FIND CONVINCING?

Problem 1 *(Student page 167)* The demonstration (if you do it "live") really *is* extremely convincing, but Raphe's explanation depends only on appearances. If you remember the "almost-invariant" of Problem 2 on page 164, you will see why appearances are important, even *very* important (because they lead to ideas), but not *enough*.

Problem 2 *(Student page 168)* Liza goes a step further than Raphe by specifying a criterion for being a parallelogram (opposite sides of equal length) and showing measurements that meet that criterion. Again (especially if you see it "live") this is *very* convincing, but it does not explain *why*, so it is not completely reassuring. Perhaps this is another "almost-invariant" like the one discovered in Problem 2 in Investigation 1.24. Interestingly, Liza's demonstration shows *two* invariants—the equality she was looking for and the completely unchanging measure of 0.68 inch for one pair of sides. That second invariant is actually a powerful clue to answer the *why* question: Why is there always a parallelogram?

Problem 3 *(Student page 168)* As B moves, $\overline{EH}$ remains parallel and congruent to $\overline{FG}$, which never even budges. Segments $\overline{EF}$ and $\overline{HG}$ move but remain parallel and congruent. Don't overlook the "obvious": points A, C, and E don't move. If you complete that triangle by sketching in segment $\overline{AC}$, you might notice that $\overline{EH}$ and $\overline{FG}$, are always parallel to $\overline{AC}$.

Problem 4 *(Student page 168)* Two pairs of measurements—of slopes, angles, or sidelengths—are the usual ways to test for parallelograms. All of these tests require four measurements. Could you do better if you looked at diagonals?

FINDING OTHER INVARIANTS

Problems 1–2 *(Student page 170)* The "midline" will always be half the length of the side to which it is parallel.

Problem 3 *(Student page 170)* This applies the same rule. The sides of the inner quadrilateral are midlines of the triangles produced by the diagonals of the kite, so if the kite has diagonals of lengths 5 and 8, the inner quadrilateral must have a perimeter of 13. All angles of the inner quadrilateral will be 90 degrees because its opposite sides are parallel to the diagonals of a kite, which are perpendicular. So the adjacent sides of the quadrilateral are perpendicular to each other. The inner quadrilateral must be a rectangle.

Problem 4 *(Student page 170)* The perimeter of the inner quadrilateral is 20. Generally, the perimeter of the parallelogram formed by the midpoints of a quadrilateral is the sum of the lengths of the diagonals of the quadrilateral.

Problem 5 *(Student page 170)*

a. The large rectangle has 8 smaller quadrilaterals nested inside. Not all of them are rectangles, but "every other one" is. The others are parallelograms of equal sides, or rhombi. Each rectangle is half the size of the one outside it, and each rhombus is also half the size of the next larger one. The perimeter is reduced by a factor of $\sqrt{2}$ from one quadrilateral to the next (or by a factor of 2 from rectangle to rectangle or rhombus to rhombus). The area is reduced by a factor of 2 from one quadrilateral to the next (or by a factor of 4 from rectangle to rectangle or rhombus to rhombus). What else do you find?

b. The perimeter of the original rectangle is $24 + 32 + 24 + 32 = 112$. Each smaller rectangle (ignoring the rhombus in between) is half the size of the previous one, and so has half the perimeter (and, incidentally, one quarter the area). So the smallest rectangle has a perimeter of 7.

Making the Right Connections

Why would Anisha bother to look for an explanation of something that doesn't change?

Problem 2 *(Student page 172)* Anisha might have noticed that moving one vertex of the original quadrilateral had no effect on one side of the interior parallelogram. In looking for an explanation of *that* thing that does not change, she might have deliberately listed all the other things that do not change. In particular, she might have noticed that the other three vertices of the original quadrilateral did not move. Those three vertices make an unchanging triangle, and the completely static side of the parallelogram *couldn't* move, because it is locked still by connecting fixed points on that triangle. Of course, to complete her idea, she would have to measure diagonal $\overline{BD}$ (which is the third side of her imaginary triangle). Anisha's approach so far is extremely smart, but with what she's already discovered, the next step (investigating the triangle by, for example, measuring $\overline{BD}$) is not a surprise.

The *Connected Geometry* module *A Matter of Scale* proves both conjectures while dealing with geometric similarity.

Problem 3 *(Student page 173)* If all Anisha knew about parallelograms was that they had opposite sides parallel, then her conjecture about lengths of triangle midlines would be useless. But Anisha was smart! If all she'd known was about parallels, she would probably not have bothered *measuring* length in the first place, because her attention would have been on the parallel lines. She might have measured slopes instead, and conjectured that the midline of a triangle is parallel to the third side. That would have suited her purposes fine (and happens also to be true).

Problem 4 *(Student page 173)* If you have not already done so, find an arrangement of A, B, C, and D that does *not* cause P and T to coincide, and experiment with P, moving it around and watching what happens to $PQRST$, and particularly to T. This may lead you to a more general conjecture than the problem asks you to find.

Solutions

Investigation 1.28

Student Pages 174–175

CAN YOU SAY MORE?

Problem 1 *(Student page 174)* The figure created by connecting the midpoints of a quadrilateral has sides parallel to the diagonals of the quadrilateral. The diagonals of a kite are perpendicular. A parallelogram with right angles is a rectangle. Therefore, the figure is a rectangle.

Problem 2 *(Student page 174)* To have all four sides of the inner parallelogram congruent, the diagonals of the outer quadrilateral must be the same length. That makes the outer figure a rectangle.

THE MIDLINE THEOREM

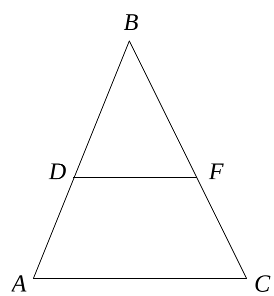
Problem 1 *(Student page 177)*

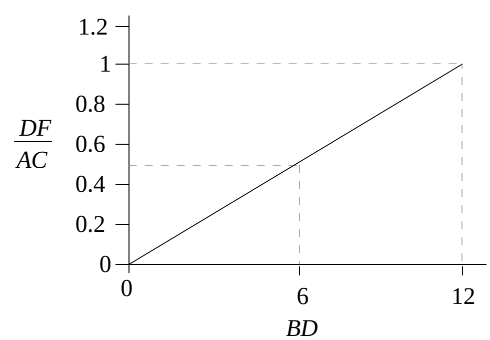
a. The argument for continuity claims that moving $\overline{DF}$ a little bit changes its length a little bit, and if it changes in length from greater than 5 to less than 5, then it must be 5 somewhere in between.

b. Without knowing something special about the situation, there is no *a priori* reason (no reason "at the outset") *for* or *against* believing that the halfway length of $\overline{DF}$ will occur at the halfway position of $\overline{DF}$. In fact, it happens to be true—the halfway position and length occur together—but that is a remarkable and important property of the relationship between these two changeable values. This property is so important that it has a name: *linearity*.

Problem 2 *(Student page 177)* There are many different appropriate choices of graphs and ways of labeling them. This particular graph was based on a triangle adjusted so that the length of $\overline{BA}$ is 12. The position of $\overline{DF}$ is expressed in terms of the length BD, which can vary from 0 to 12. Clearly, the ratio $\frac{DF}{AC}$ can vary only between 0 and 1.

The important thing to notice in any graph that shows how $\overline{DF}$'s position relates to $\frac{DF}{AC}$ is that the graphs are straight line—the relationship between these two quantities is linear. The linearity guarantees that when D is halfway (at 6), $\frac{DF}{AC}$ is also halfway (at 0.5).



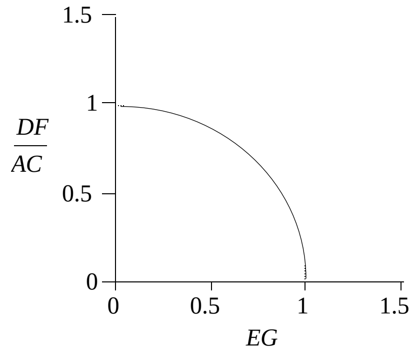


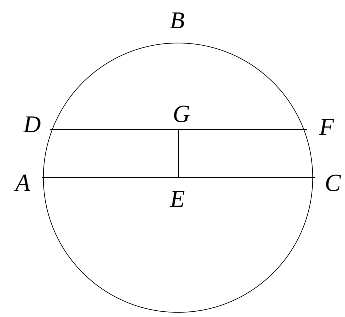
Instead of labeling the horizontal axis of the graph with *length* (which leaves us to figure out what is halfway), we might have performed that computation in advance and labeled it with the fraction we're interested in, $\frac{BD}{BA}$, which, like $\frac{DF}{AC}$, also varies from 0 to 1. In that case, the graph compares these two ratios and shows them to be equal!

Problem 3 *(Student page 177)*

a. Again, there are many sensible ways of drawing the graph. This particular graph is based on a circle whose radius is 1 (that is, $AC = 2$). Because the radius is 1, the numbers for the horizontal axis work out the same, whether we use distance EG or ratio $\frac{EG}{EB}$ (where B is taken to be the "top" of the semicircle, the farthest up that $\overline{DF}$ can go).

E is the center of the circle, and G is the midpoint of $\overline{DF}$, so EG is the distance from $\overline{AC}$ to $\overline{DF}$. The longest $\overline{DF}$ can get is 2, which makes $\frac{DF}{AC} = 1$. The shortest $\overline{DF}$ can get is zero, when EG is at its longest.





b. "Halfway up" occurs when EG is 0.5. At that point, $\frac{DF}{AC}$ is clearly more than 0.5. The relationship between $\frac{DF}{AC}$ and EG is *not* linear, and the graph is *not* a straight line. Thinking about the geometry of it, this is no surprise: $\frac{DF}{AC}$ was half at the halfway-up point with a *triangle*, and a circle bulges out more, so DF stays longer for longer.

What is the area formula for a trapezoid?

Problem 4 *(Student page 177)* The largest value of EG is AB, and the smallest is DC. Experimenting shows that the relationship between $\frac{EG}{AB}$ and $\frac{DE}{DA}$ is again linear (though not, in this case, an equality). So, when $\overline{EG}$ is halfway between $\overline{DC}$ and $\overline{AB}$, then its length will be halfway their two lengths: $\frac{AB+DC}{2}$.